





















# ELEMENTS

OF

# GEOMETRY AND TRIGONOMETRY,

FROM THE WORKS OF

A. M. LEGENDRE.

ADAPTED TO THE COURSE OF MATHEMATICAL INSTRUCTION IN  
THE UNITED STATES,

BY CHARLES DAVIES, LL.D.,

AUTHOR OF ARITHMETIC, ALGEBRA, PRACTICAL MATHEMATICS FOR PRACTICAL MEN,  
ELEMENTS OF DESCRIPTIVE AND OF ANALYTICAL GEOMETRY, ELEMENTS  
OF DIFFERENTIAL AND INTEGRAL CALCULUS, AND SHADES,  
SHADOWS, AND PERSPECTIVE.

NEW YORK :

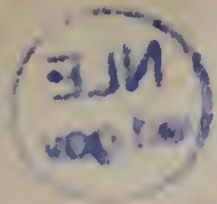
BARNES & BURR, PUBLISHERS, 51 & 53 JOHN ST.

CHICAGO: GEORGE SHERWOOD, 118 LAKE ST.

CINCINNATI: RICKEY AND CARROLL

ST. LOUIS: KEITH AND WOODS.

1864.



# Davies' Course of Mathematics.

**Davies' Primary Arithmetic AND Table-Book**—Designed for Beginners; containing the elementary tables of Addition, Subtraction, Multiplication, Division, and Denominate Numbers; with a large number of easy and practical questions, both mental and written.

**Davies' First Lessons in Arithmetic**—Combining the Oral Method with the Method of Teaching the Combinations of Figures by Sight.

**Davies' Intellectual Arithmetic**—An Analysis of the Science of Numbers, with especial reference to Mental Training and Development.

**Davies' New School Arithmetic**—Analytical and Practical.

**Key to Davies' New School Arithmetic.**

**Davies' Grammar of Arithmetic**—An Analysis of the Language of Numbers and the Science of Figures.

**Davies' New University Arithmetic**—Embracing the Science of Numbers, and their Applications according to the most Improved Methods of Analysis and Cancellation.

**Key to Davies' New University Arithmetic.**

**Davies' Elementary Algebra**—Embracing the First Principles of the Science.

**Key to Davies' Elementary Algebra.**

**Davies' Elementary Geometry AND Trigonometry**—With Applications in Mensuration.

**Davies' Practical Mathematics**—With Drawing and Mensuration applied to the Mechanic Arts.

**Davies' University Algebra**—Embracing a Logical Development of the Science, with graded examples.

**Davies' Bourdon's Algebra**—Including Sturm's and Horner's Theorems, and practical examples.

**Davies' Legendre's Geometry and Trigonometry**—Revised and adapted to the course of Mathematical Instruction in the United States.

**Davies' Elements of Surveying AND Navigation**—Containing descriptions of the Instruments and necessary Tables.

**Davies' Analytical Geometry**—Embracing the Equations of the Point, the Straight Line, the Conic Sections, and Surfaces of the first and second order.

**Davies' Differential AND Integral Calculus.**

**Davies' Descriptive Geometry**—With its application to Spherical Trigonometry, Spherical Projections, and Warped Surfaces.

**Davies' Shades, Shadows, AND Perspective.**

**Davies' Logic and Utility of Mathematics**—With the best methods of Instruction Explained and Illustrated.

**Davies' and Peck's Mathematical Dictionary and Cyclopædia of Mathematical Science**—Comprising Definitions of all the terms employed in Mathematics—an Analysis of each Branch, and of the whole, as forming a single Science.

---

ENTERED, according to Act of Congress, in the year one thousand eight hundred and sixty-two, by CHARLES DAVIES, in the Clerk's Office of the District Court of the United States for the Southern District of New York.

---



## P R E F A C E .

---

OF the various Treatises on Elementary Geometry which have appeared during the present century, that of M. LEGENDRE stands preëminent. Its peculiar merits have won for it not only a European reputation, but have also caused it to be selected as the basis of many of the best works on the subject that have been published in this country.

In the original Treatise of LEGENDRE, the propositions are not enunciated in general terms, but by means of the diagrams employed in their demonstration. This departure from the method of EUCLID is much to be regretted. The propositions of Geometry are general truths, and ought to be stated in general terms, without reference to particular diagrams. In the following work, each proposition is first enunciated in general terms, and afterwards, with reference to a particular figure, that figure being taken to represent any one of the class to which it belongs. By this arrangement, the difficulty experienced by beginners in comprehending abstract truths, is lessened, without in any manner impairing the generality of the truths evolved.

The term *solid*, used not only by LEGENDRE, but by many other authors, to denote a limited portion of space, seems calculated to introduce the foreign idea of matter

into a science, which deals only with the abstract properties and relations of figured space. The term *volume*, has been introduced in its place, under the belief that it corresponds more exactly to the idea intended. Many other departures have been made from the original text, the value and utility of which have been made manifest in the practical tests to which the work has been subjected.

In the present Edition, numerous changes have been made, both in the Geometry and in the Trigonometry. The definitions have been carefully revised—the demonstrations have been harmonized, and, in many instances, abbreviated—the principal object being to simplify the subject as much as possible, without departing from the general plan. These changes are due to Professor Peck, of the Department of Pure Mathematics and Astronomy in Columbia College. For his aid, in giving to the work its present permanent form, I tender him my grateful acknowledgements.

CHARLES DAVIES.

COLUMBIA COLLEGE,  
NEW YORK, April, 1862.



# CONTENTS.

---

## GEOMETRY.

	PAGE.
INTRODUCTION, .....	9
BOOK I.	
Definitions, .....	13
Propositions, .....	20
BOOK II.	
Ratios and Proportions, .....	50
BOOK III.	
The Circle, and the Measurement of Angles, .....	59
Problems relating to the First and Third Books, .....	82
BOOK IV.	
Proportions of Figures—Measurement of Areas, .....	93
Problems relating to the Fourth Book, .....	129
BOOK V.	
Regular Polygons—Measurement of the Circle, .....	136
BOOK VI.	
Planes, and Polyedral Angles, .....	157
BOOK VII.	
Polyedrons, .....	178

## CONTENTS.

## BOOK VIII.

	PAGE.
Cylinder, Cone, and Sphere, .....	210

## BOOK IX.

Spherical Geometry, .....	235
---------------------------	-----

## PLANE TRIGONOMETRY.

## INTRODUCTION.

Definition of Logarithms, .....	3
Rules for Characteristics, .....	4
General Principles, .....	5
Table of Logarithms, .....	7
Manner of Using the Table, .....	8
Multiplication by Logarithms, .....	11
Division by Logarithms, .....	12
Arithmetical Complement, .....	13
Raising to Powers by Logarithms, .....	15
Extraction of Roots by Logarithms, .....	16

## PLANE TRIGONOMETRY.

Plane Trigonometry Defined, .....	17
Functions of the Arc, .....	18-22
Table of Natural Sines, .....	22
Table of Logarithmic Sines, .....	22
Use of the Table, .....	23-27
Solution of Right-angled Triangles, .....	27-35
Solution of Oblique-angled Triangles, .....	36-47
Problems of Application, .....	48

## ANALYTICAL TRIGONOMETRY.

Analytical Trigonometry Defined, .....	51
Definitions and General Principles, .....	51-54
Rules for Signs of the Functions, .....	54

# CONTENTS.

vii

	PAGE.
Limiting value of Circular Functions, .....	55
Relations of Circular Functions, .....	57-59
Functions of Negative Arcs, .....	60-62
Particular values of Certain Functions, .....	63
Formulas of Relation between Functions and Arcs,.....	64-66
Functions of Double and Half Arcs, .....	67
Additional Formulas, .....	68-70
Method of Computing a Table of Natural Sines,.....	71

## SPHERICAL TRIGONOMETRY.

Spherical Trigonometry Defined, .....	73
General Principles, .....	73
Formulas for Right-angled Triangles, .....	74-76
Napier's Circular Parts, .....	77
Solution of Right-angled Spherical Triangles, .....	80-83
Quadrantal Triangles, .....	84
Formulas for Oblique-angled Triangles,.....	85-92
Solution of Oblique-angled Triangles, .....	92-104

## MENSURATION.

Mensuration Defined, .....	105
The Area of a Parallelogram,.....	106
The Area of a Triangle, .....	106
Formula for the Sine of Half an Angle,.....	108
Area of a Trapezoid, .....	112
Area of a Quadrilateral, .....	112
Area of a Polygon, .....	113
Area of a Regular Polygon, .....	114
To find the Circumference of a Circle, .....	116
To find the Diameter of a Circle, .....	116
To find the length of an Arc, .....	117
Area of a Circle, .....	117
Area of a Sector, .....	118
Area of a Segment, .....	118
Area of a Circular Ring,.....	119



	PAGE.
Area of the Surface of a Prism, .....	120
Area of the Surface of a Pyramid, .....	120
Area of the Frustum of a Cone, .....	121
Area of the Surface of a Sphere, .....	122
Area of a Zone, .....	122
Area of a Spherical Polygon, .....	123
Volume of a Prism, .....	124
Volume of a Pyramid, .....	124
Volume of the Frustum of a Pyramid, .....	125
Volume of a Sphere, .....	126
Volume of a Wedge, .....	127
Volume of a Prismoid, .....	128
Volumes of Regular Polyedrons, .....	132

# ELEMENTS OF GEOMETRY.

---

## INTRODUCTION.

1. QUANTITY is anything that can be increased, diminished, and measured.

To measure a thing is to find how many times it contains some other thing of the same kind, taken as a *standard*. The assumed standard is called the *unit of measure*.

2. Since the unit of measure is of the same kind as the thing measured, there are as many kinds of units as there are species of quantity. In Geometry, there are four species of quantity, called GEOMETRICAL MAGNITUDES, viz., LINES, SURFACES, VOLUMES, and ANGLES: hence, there are four kinds of units of measure, viz., *Units of Length*, *Units of Surface*, *Units of Volume*, and *Units of Angular Measure*.

3. GEOMETRY is that branch of Mathematics which treats of the properties and relations of Geometrical Magnitudes.

4. In Geometry, the quantities considered are generally represented by pictorial symbols. The operations to be performed upon them, and the relations between them, are indicated by signs, as in Analysis.

The following are the principal signs employed :

The *Sign of Addition*,  $+$ , called *plus* :

Thus,  $A + B$ , indicates that  $B$  is to be added to  $A$ .

The *Sign of Subtraction*,  $-$ , called *minus* :

Thus,  $A - B$ , indicates that  $B$  is to be subtracted from  $A$ .

The *Sign of Multiplication*,  $\times$  :

Thus,  $A \times B$ , indicates that  $A$  is to be multiplied by  $B$ .

The *Sign of Division*,  $\div$  :

Thus,  $A \div B$ , or,  $\frac{A}{B}$ , indicates that  $A$  is to be divided by  $B$ .

The *Exponential Sign* :

Thus,  $A^3$ , indicates that  $A$  is to be taken three times as a factor, or raised to the third power.

The *Radical Sign*,  $\sqrt{\phantom{x}}$  :

Thus,  $\sqrt{A}$ ,  $\sqrt[3]{B}$ , indicate that the square root of  $A$ , and the cube root of  $B$ , are to be taken.

When a compound quantity is to be operated upon as a single quantity, its parts are connected by a vinculum or by a parenthesis :

Thus,  $\overline{A + B} \times C$ , indicates that the sum of  $A$  and  $B$  is to be multiplied by  $C$ ; and  $(A + B) \div C$ , indicates that the sum of  $A$  and  $B$  is to be divided by  $C$ .

A number written before a quantity, shows how many times it is to be taken.

Thus,  $3(A + B)$ , indicates that the sum of  $A$  and  $B$  is to be taken three times.

The *Sign of Equality*,  $=$  :

Thus,  $A = B + C$ , indicates that  $A$  is equal to the sum of  $B$  and  $C$ .



The expression,  $A = B + C$ , is called an equation. The part on the left of the sign of equality, is called the *first member*; that on the right, the *second member*.

The *Sign of Inequality*,  $<$  :

Thus,  $\sqrt{A} < \sqrt[3]{B}$ , indicates that the square root of  $A$  is less than the cube root of  $B$ . The opening of the sign is towards the greater quantity.

The sign,  $\therefore$  is used as an abbreviation of the word *hence*, or *consequently*.

5. The general truths of Geometry are deduced by a course of logical reasoning, the premises being definitions and principles previously established. The course of reasoning employed in establishing any truth or principle, is called a *demonstration*.

6. A THEOREM is a truth requiring demonstration.

7. An AXIOM is a self-evident truth.

8. A PROBLEM is a question requiring a solution.

9. A POSTULATE is a problem whose solution is self-evident.

Theorems, Axioms, Problems, and Postulates, are all called *Propositions*.

10. A LEMMA is an auxiliary proposition.

11. A COROLLARY is an obvious consequence of one or more propositions.

12. A SCHOLIUM is a remark made upon one or more propositions, with reference to their connection, their use, their extent, or their limitation.

13. An HYPOTHESIS is a supposition made, either in the statement of a proposition, or in the course of a demonstration.

14. Two magnitudes are *equal*, when they are equal in measure.

When they may be so placed as to coincide throughout their whole extent, they are *equal in all their parts*.



# ELEMENTS OF GEOMETRY.

---

## BOOK I.

### ELEMENTARY PRINCIPLES.

#### DEFINITIONS.

1. GEOMETRY is that branch of Mathematics which treats of the properties and relations of Geometrical Magnitudes.

2. A POINT is that which has position, but not magnitude.

3. A LINE is that which has length, but neither breadth nor thickness.

Lines are divided into two classes, *straight* and *curved*.

4. A STRAIGHT LINE is one which does not change its direction at any point.

5. A CURVED LINE is one which changes its direction at every point.

The word *line*, alone, is used for *straight line*; and the word *curve*, alone, for *curved line*.

6. A line made up of straight lines, not lying in the same direction, is called a *broken line*.

7. A SURFACE is that which has length and breadth, without thickness.

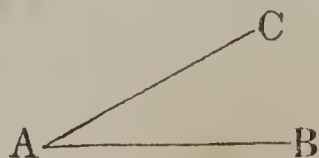
Surfaces are divided into two classes, *plane* and *curved surfaces*.

8. A **PLANE** is a surface, such, that if any two of its points be joined by a straight line, that line will lie wholly in the surface.

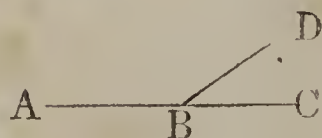
9. A **CURVED SURFACE** is a surface which is neither a plane nor composed of planes.

10. A **PLANE ANGLE** is the amount of divergence of two lines lying in the same plane.

Thus, the amount of divergence of the lines  $AB$  and  $AC$ , is an angle. The lines  $AB$  and  $AC$  are called *sides*, and their common point  $A$ , is called the *vertex*. An angle is designated by naming its sides, or sometimes by simply naming its vertex; thus, the above is called the angle  $BAC$ , or simply, the angle  $A$ .



11. When one straight line meets another the two angles which they form are called *adjacent angles*. Thus, the angles  $ABD$  and  $DBC$  are adjacent.



12. A **RIGHT ANGLE** is formed by one straight line meeting another so as to make the adjacent angles *equal*. The first line is then said to be *perpendicular* to the second.



13. An **OBLIQUE ANGLE** is formed by one straight line meeting another so as to make the adjacent angles *unequal*.



Oblique angles are subdivided into two classes, *acute angles*, and *obtuse angles*.

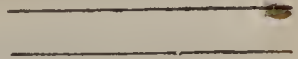
14. An **ACUTE ANGLE** is less than a right angle



15. An OBTUSE ANGLE is greater than a right angle.



16. Two straight lines are *parallel*, when they lie in the same plane and cannot meet, how far soever, either way, both may be produced. They then have the *same direction*.



17. A PLANE FIGURE is a portion of a plane bounded by lines, either straight or curved.

18. A POLYGON is a plane figure bounded by straight lines.

The bounding lines are called *sides* of the polygon. The broken line, made up of all the sides of the polygon, is called the *perimeter* of the polygon. The angles formed by the sides, are called *angles* of the polygon.

19. Polygons are classified according to the number of their sides or angles.

A Polygon of three sides is called a *triangle*; one of four sides, a *quadrilateral*; one of five sides, a *pentagon*; one of six sides, a *hexagon*; one of seven sides, a *heptagon*; one of eight sides, an *octagon*; one of ten sides, a *decagon*; one of twelve sides, a *dodecagon*, &c.

20. An EQUILATERAL POLYGON, is one whose sides are all equal.

An EQUIANGULAR POLYGON, is one whose angles are all equal.

A REGULAR POLYGON, is one which is both equilateral and equiangular.

21. Two polygons are *equilateral*, or *mutually equilateral*, when their sides, taken in the same order, are equal, each to each: that is, following their perimeters in the same



direction, the first side of the one is equal to the first side of the other, the second side of the one, to the second side of the other, and so on.

22. Two polygons are *equiangular*, or *mutually equiangular*, when their angles, taken in the same order, are equal, each to each.

23. A **DIAGONAL** of a polygon is a line joining the vertices of two angles, not consecutive.

24. A **BASE** of a polygon is any one of its sides on which the polygon is supposed to stand.

25. Triangles may be classified with reference either to their sides, or their angles.

When classified with reference to their sides, there are two classes: *scalene* and *isosceles*.

1st. A **SCALED TRIANGLE** is one which has no two of its sides equal.



2d. An **ISOSCELES TRIANGLE** is one which has two of its sides equal.

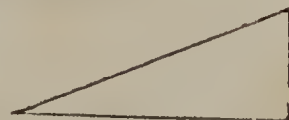


When all of the sides are equal, the triangle is **EQUILATERAL**.



When classified with reference to their angles, there are two classes: *right-angled* and *oblique-angled*.

1st. A **RIGHT-ANGLED TRIANGLE** is one that has one right angle.



The side opposite the right angle, is called the *hypotenuse*.

2d. An **OBLIQUE-ANGLED TRIANGLE** is one whose angles are all oblique.





If one angle of an oblique-angled triangle is obtuse, the triangle is said to be OBTUSE-ANGLED. If all of the angles are acute, the triangle is said to be ACUTE-ANGLED.

26. Quadrilaterals are classified with reference to the relative directions of their sides. There are then two classes: the *first class* embraces those which have no two sides parallel; the *second class* embraces those which have two sides parallel.

Quadrilaterals of the first class, are called *trapeziums*.

Quadrilaterals of the second class, are divided into two species: *trapezoids* and *parallelograms*.

27. A TRAPEZOID is a quadrilateral which has only two of its sides parallel.



28. A PARALLELOGRAM is a quadrilateral which has its opposite sides parallel, two and two.

There are two varieties of parallelograms: *rectangles* and *rhomboids*.

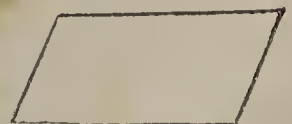
1st. A RECTANGLE is a parallelogram whose angles are all right angles.



A SQUARE is an equilateral rectangle.



2d. A RHOMBOID is a parallelogram whose angles are all oblique.



A RHOMBUS is an equilateral rhomboid.



## AXIOMS.

1. Things which are equal to the same thing, are equal to each other.
2. If equals be added to equals, the sums will be equal.
3. If equals be subtracted from equals, the remainders will be equal.
4. If equals be added to unequals, the sums will be unequal.
5. If equals be subtracted from unequals, the remainders will be unequal.
6. If equals be multiplied by equals, the products will be equal.
7. If equals be divided by equals, the quotients will be equal.
8. The whole is greater than any of its parts.
9. The whole is equal to the sum of all its parts.
10. All right angles are equal.
11. Only one straight line can be drawn between two points.
12. The shortest distance between any two points is measured on the straight line which joins them.
13. Through the same point, only one line can be drawn parallel to a given line.

## POSTULATES.

1. A straight line can be drawn between any two points.
2. A straight line may be prolonged to any length.
3. If two lines are unequal, the length of the less may be laid off on the greater

4. A line may be bisected; that is, divided into two equal parts.

5. An angle may be bisected.

6. A perpendicular may be drawn to a given line, either from a point without, or from a point on the line.

7. A line may be drawn, making with a given line an angle equal to a given angle.

8. A line may be drawn through a given point, parallel to a given line.

#### NOTE.

In making references, the following abbreviations are employed, viz.: A. for Axiom; B. for Book; C. for Corollary; D. for Definition; I. for Introduction; P. for Proposition; Prob. for Problem; Post. for Postulate; and S. for Scholium. In referring to the same Book, the number of the Book is *not* given; in referring to any other Book, the number of the Book is given.

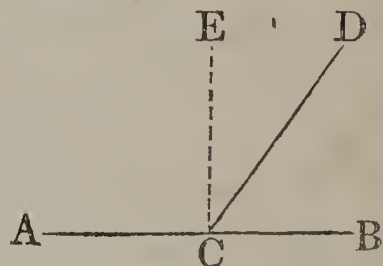


## PROPOSITION I. THEOREM.

*If a straight line meet another straight line, the sum of the adjacent angles will be equal to two right angles.*

Let  $DC$  meet  $AB$  at  $C$ :  
then will the sum of the angles  $DCA$  and  $DCB$  be equal to two right angles.

At  $C$ , let  $CE$  be drawn perpendicular to  $AB$  (Post. 6); then, by definition (D. 12), the angles  $ECA$  and  $ECB$  will both be right angles, and consequently, their sum will be equal to *two right angles*.



The angle  $DCA$  is equal to the sum of the angles  $ECA$  and  $ECD$  (A. 9); hence,

$$DCA + DCB = ECA + ECD + DCB;$$

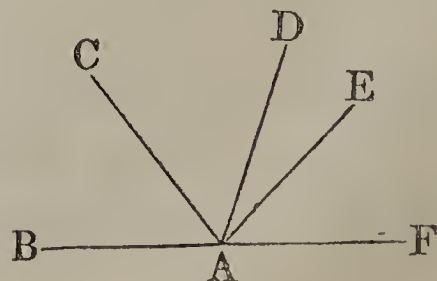
But,  $ECD + DCB$  is equal to  $ECB$  (A. 9); hence,

$$DCA + DCB = ECA + ECB.$$

The sum of the angles  $ECA$  and  $ECB$ , is equal to two right angles; consequently, its equal, that is, the sum of the angles  $DCA$  and  $DCB$ , must also be equal to two right angles; *which was to be proved*.

*Cor. 1.* If one of the angles  $DCA$ ,  $DCB$ , is a right angle, the other must also be a right angle.

*Cor. 2.* The sum of the angles  $BAC$ ,  $CAD$ ,  $DAE$ ,  $EAF$ , formed about a given point on the same side of a straight line  $BF$ , is equal to two right angles. For, their sum is equal to



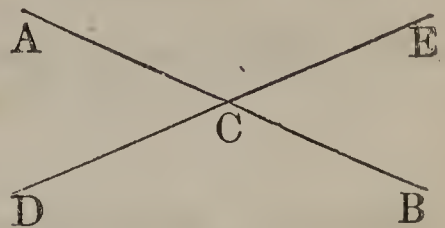


the sum of the angles  $EAB$  and  $EAF$ ; which, from the proposition just demonstrated, is equal to two right angles.

## DEFINITIONS.

If two straight lines intersect each other, they form four angles about the point of intersection, which have received different names, with respect to each other.

1°. ADJACENT ANGLES are those which lie on the same side of one line, and on opposite sides of the other; thus,  $ACE$  and  $ECB$ , or  $ACE$  and  $ACD$ , are adjacent angles.

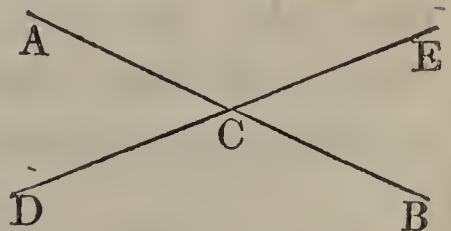


2°. OPPOSITE, or VERTICAL ANGLES, are those which lie on opposite sides of both lines; thus,  $ACE$  and  $DCB$ , or  $ACD$  and  $ECB$ , are opposite angles. From the proposition just demonstrated, the sum of any two adjacent angles is equal to two right angles.

## PROPOSITION II. THEOREM.

*If two straight lines intersect each other, the opposite or vertical angles will be equal.*

Let  $AB$  and  $DE$  intersect at  $C$ : then will the opposite or vertical angles be equal.

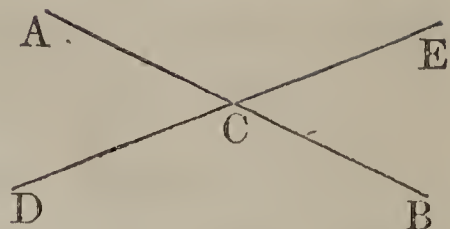


The sum of the adjacent angles  $ACE$  and  $ACD$ , is equal to two right angles (P. I.): the sum of the adjacent angles  $ACE$  and  $ECB$ , is also equal to two right angles. But things which are equal to the same thing, are equal to each other (A. 1); hence,

$$ACE + ACD = ACE + ECB;$$

Taking from both the common angle  $ACE$  (A. 3), there remains,

$$ACD = ECB.$$



In like manner, we find,

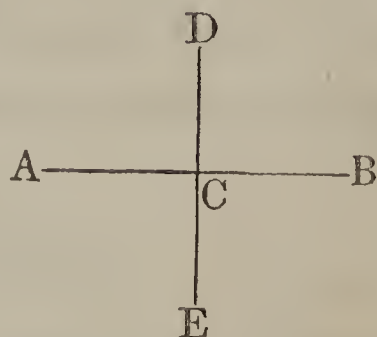
$$ACD + ACE = ACD + DCB;$$

and, taking away the common angle  $ACD$ , we have,

$$ACE = DCB.$$

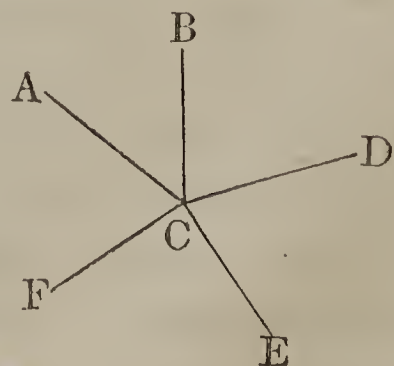
Hence, *the proposition is proved.*

*Cor. 1.* If one of the angles about  $C$  is a right angle, all of the others will be right angles also. For, (P. I., C. 1), each of its adjacent angles will be a right angle; and from the proposition just demonstrated, its opposite angle will also be a right angle.



*Cor. 2.* If one line  $DE$ , is perpendicular to another  $AB$ , then will the second line  $AB$  be perpendicular to the first  $DE$ . For, the angles  $DCA$  and  $DCB$  are right angles, by definition (D. 12); and from what has just been proved, the angles  $ACE$  and  $BCE$  are also right angles. Hence, the two lines are mutually perpendicular to each other.

*Cor. 3.* The sum of all the angles  $ACB$ ,  $BCD$ ,  $DCE$ ,  $ECF$ ,  $FCA$ , that can be formed about a point, is equal to four right angles.

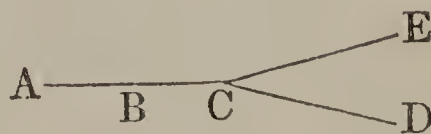


For, if two lines be drawn through the point, mutually perpendicular to each other, the sum of the angles which they form will be equal to four right angles, and it will also be equal to the sum of the given angles (A. 9). Hence, the sum of the given angles is equal to four right angles.

### PROPOSITION III. THEOREM.

*If two straight lines have two points in common, they will coincide throughout their whole extent, and form one and the same line.*

Let  $A$  and  $B$  be two points common to two lines: then will the lines coincide throughout.



Between  $A$  and  $B$  they must coincide (A. 11). Suppose, now, that they begin to separate at some point  $C$ , beyond  $AB$ , the one becoming  $ACE$ , and the other  $ACD$ . If the lines do separate at  $C$ , one or the other must change direction at this point; but this is contradictory to the definition of a straight line (D. 4): hence, the supposition that they separate at any point is absurd. They must, therefore, coincide throughout; *which was to be proved.*

*Cor.* Two straight lines can intersect in only one point.

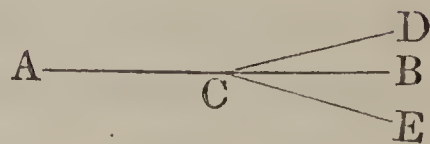
NOTE.—The method of demonstration employed above, is called the *reductio ad absurdum*. It consists in assuming an hypothesis which is the contradictory of the proposition to be proved, and then continuing the reasoning until the assumed hypothesis is shown to be false. Its contradictory is thus proved to be true. This method of demonstration is often used in Geometry.



## PROPOSITION IV. THEOREM.

*If a straight line meet two other straight lines at a common point, making the sum of the contiguous angles equal to two right angles, the two lines met will form one and the same straight line.*

Let  $DC$  meet  $AC$  and  $BC$  at  $C$ , making the sum of the angles  $DCA$  and  $DCB$  equal to two right angles: then will  $CB$  be the prolongation of  $AC$ .



For, if not, suppose  $CE$  to be the prolongation of  $AC$ , then will the sum of the angles  $DCA$  and  $DCE$  be equal to two right angles (P. I.): We shall, consequently, have (A. 1),

$$DCA + DCB = DCA + DCE;$$

Taking from both the common angle  $DCA$ , there remains,

$$DCB = DCE,$$

which is impossible, since a part cannot be equal to the whole (A. 8). Hence,  $CB$  must be the prolongation of  $AC$ ; which was to be proved.

## PROPOSITION V. THEOREM.

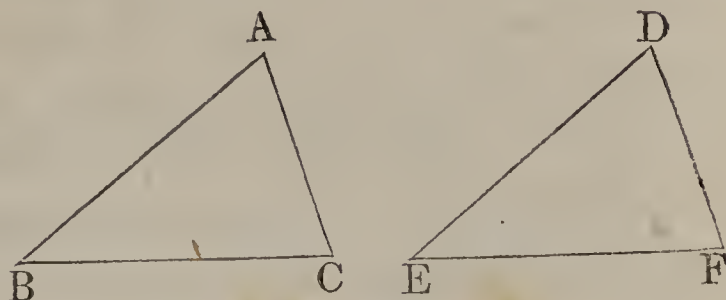
*If two triangles have two sides and the included angle of the one equal to two sides and the included angle of the other, each to each, the triangles will be equal in all their parts.*

In the triangles  $ABC$  and  $DEF$ , let  $AB$  be equal



to  $DE$ ,  $AC$  to  $DF$ , and the angle  $A$  to the angle  $D$ : then will the triangles be equal in all their parts.

For, let  $ABC$  be applied to  $DEF$ , in such a manner that the angle  $A$  shall coincide with the angle  $D$ , the side  $AB$  taking the direction  $DE$ , and

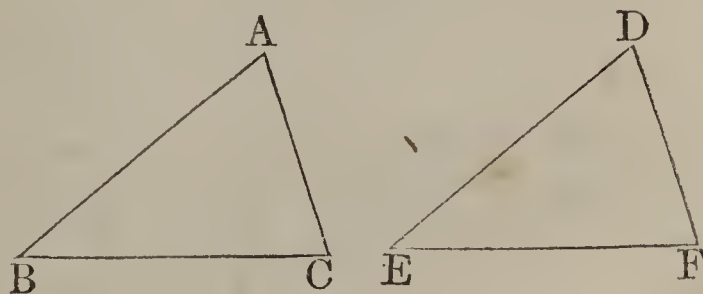


the side  $AC$  the direction  $DF$ . Then, because  $AB$  is equal to  $DE$ , the vertex  $B$  will coincide with the vertex  $E$ ; and because  $AC$  is equal to  $DF$ , the vertex  $C$  will coincide with the vertex  $F$ ; consequently, the side  $BC$  will coincide with the side  $EF$  (A. 11). The two triangles, therefore, coincide throughout, and are consequently equal in all their parts (I., D. 14); *which was to be proved.*

#### PROPOSITION VI. THEOREM.

*If two triangles have two angles and the included side of the one equal to two angles and the included side of the other, each to each, the triangles will be equal in all their parts.*

In the triangles  $ABC$  and  $DEF$ , let the angle  $B$  be equal to the angle  $E$ , the angle  $C$  to the angle  $F$ , and the side  $BC$  to the side  $EF$ : then will the triangles be equal in all their parts.



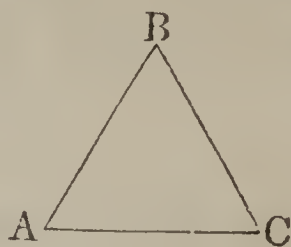
For, let  $ABC$  be applied to  $DEF$  in such a manner that the angle  $B$  shall coincide with the angle  $E$ , the side

$BC$  taking the direction  $EF$ , and the side  $BA$  the direction  $ED$ . Then, because  $BC$  is equal to  $EF$ , the vertex  $C$  will coincide with the vertex  $F$ ; and because the angle  $C$  is equal to the angle  $F$ , the side  $CA$  will take the direction  $FD$ . Now, the vertex  $A$  being at the same time on the lines  $ED$  and  $FD$ , it must be at their intersection  $D$  (P. III., C.) : hence, the triangles coincide throughout; and are therefore equal in all their parts (I., D. 14) ; *which was to be proved.*

PROPOSITION VII. THEOREM.

*The sum of any two sides of a triangle is greater than the third side.*

Let  $ABC$  be a triangle: then will the sum of any two sides, as  $AB$ ,  $BC$ , be greater than the third side  $AC$ .



For, the distance from  $A$  to  $C$ , measured on any broken line  $AB$ ,  $BC$ , is greater than the distance measured on the straight line  $AC$  (A. 12) : hence, the sum of  $AB$  and  $BC$  is greater than  $AC$  ; *which was to be proved.*

*Cor.* If from both members of the inequality,

$$AC < AB + BC,$$

we take away either of the sides  $AB$ ,  $BC$ , as  $BC$ , for example, there will remain (A. 5),

$$AC - BC < AB ;$$

that is, *the difference between any two sides of a triangle is less than the third side.*

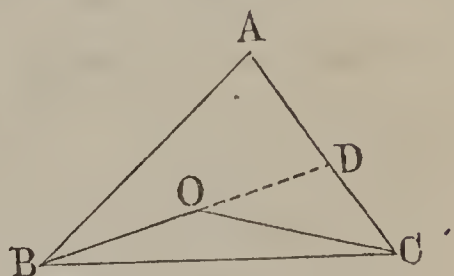
*Scholium.* In order that any three given lines may re-

present the sides of a triangle, the sum of any two must be greater than the third, and the difference of any two must be less than the third.

## PROPOSITION VIII. THEOREM.

*If from any point within a triangle two straight lines be drawn to the extremities of any side, their sum will be less than that of the two remaining sides of the triangle.*

Let  $O$  be any point within the triangle  $BAC$ , and let the lines  $OB$ ,  $OC$ , be drawn to the extremities of any side, as  $BC$ : then will the sum of  $BO$  and  $OC$  be less than the sum of the sides  $BA$  and  $AC$ .



Prolong one of the lines, as  $BO$ , till it meets the side  $AC$  in  $D$ ; then, from Prop. VII., we shall have,

$$OC < OD + DC;$$

adding  $BO$  to both members of this inequality, recollecting that the sum of  $BO$  and  $OD$  is equal to  $BD$ , we have (A. 4),

$$BO + OC < BD + DC.$$

From the triangle  $BAD$ , we have (P. VII.),

$$BD < BA + AD;$$

adding  $DC$  to both members of this inequality, recollecting that the sum of  $AD$  and  $DC$  is equal to  $AC$ , we have,

$$BD + DC < BA + AC.$$

But it was shown that  $BO + OC$  is less than  $BD + DC$ ; still more, then, is  $BO + OC$  less than  $BA + AC$ ; which was to be proved.



## PROPOSITION IX. THEOREM.

*If two triangles have two sides of the one equal to two sides of the other, each to each, and the included angles unequal, the third sides will be unequal; and the greater side will belong to the triangle which has the greater included angle.*

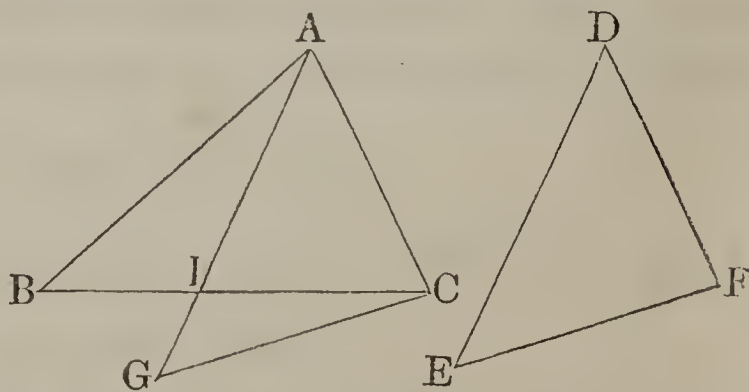
In the triangles  $BAC$  and  $DEF$ , let  $AB$  be equal to  $DE$ ,  $AC$  to  $DF$ , and the angle  $A$  greater than the angle  $D$ : then will  $BC$  be greater than  $EF$ .

Let the line  $AG$  be drawn, making the angle  $CAG$  equal to the angle  $D$  (Post. 7); make  $AG$  equal to  $DE$ , and draw  $GC$ . Then will the triangles  $AGC$  and  $DEF$  have two sides and the included angle of the one equal to two sides and the included angle of the other, each to each; consequently,  $GC$  is equal to  $EF$  (P. V.).

Now, the point  $G$  may be without the triangle  $ABC$ , it may be on the side  $BC$ , or it may be within the triangle  $ABC$ . Each case will be considered separately.

1°. When  $G$  is without the triangle  $ABC$ .

In the triangles  $GIC$  and  $AIB$ , we have, (P. VII.),



$$GI + IC > GC, \quad \text{and} \quad BI + IA > AB;$$

whence, by addition, recollecting that the sum of  $BI$  and  $IC$  is equal to  $BC$ , and the sum of  $GI$  and  $IA$ , to  $GA$ , we have,

$$AG + BC > AB + GC$$

Or, since  $AG = AB$ , and  $GC = EF$ , we have,

$$AB + BC > AB + EF.$$

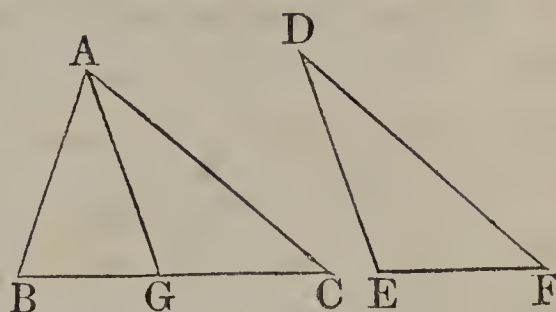
Taking away the common part  $AB$ , there remains (A. 5),

$$BC > EF.$$

2°. When  $G$  is on  $BC$ .

In this case, it is obvious that  $GC$  is less than  $BC$ ; or, since  $GC = EF$ , we have,

$$BC > EF.$$



3°. When  $G$  is within the triangle  $ABC$ .

From Proposition VIII., we have,

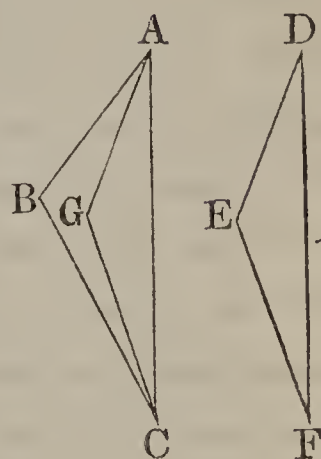
$$BA + BC > GA + GC;$$

or, since  $GA = BA$ , and  $GC = EF$ , we have,

$$BA + BC > BA + EF.$$

Taking away the common part  $AB$ , there remains,

$$BC > EF.$$



Hence, in each case,  $BC$  is greater than  $EF$ ; which was to be proved.

*Conversely*: If in two triangles  $ABC$  and  $DEF$ , the side  $AB$  is equal to the side  $DE$ , the side  $AC$  to  $DF$ , and  $BC$  greater than  $EF$ , then will the angle  $BAC$  be greater than the angle  $EDF$ .

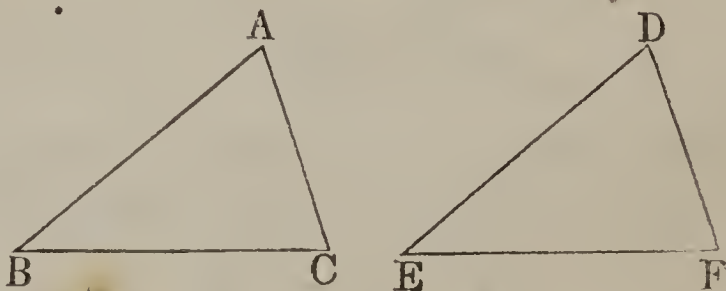
For, if not,  $BAC$  must either be equal to, or less than,  $EDF$ . In the former case,  $BC$  would be equal to  $EF$  (P. V.), and in the latter case,  $BC$  would be less than  $EF$ ; either of which would be contrary to the hypothesis hence,  $BAC$  must be greater than  $EDF$ .

## PROPOSITION X. THEOREM.

*If two triangles have the three sides of the one equal to the three sides of the other, each to each, the triangles will be equal in all their parts.*

In the triangles  $ABC$  and  $DEF$ , let  $AB$  be equal to  $DE$ ,  $AC$  to  $DF$ , and  $BC$  to  $EF$ : then will the triangles be equal in all their parts.

For, since the sides  $AB$ ,  $AC$ , are equal to  $DE$ ,  $DF$ , each to each, if the angle  $A$  were greater than  $D$ , it would follow, by the last Proposition, that the side



$BC$  would be greater than  $EF$ ; and if the angle  $A$  were less than  $D$ , the side  $BC$  would be less than  $EF$ . But  $BC$  is equal to  $EF$ , by hypothesis; therefore, the angle  $A$  can neither be greater nor less than  $D$ : hence, it must be equal to it. The two triangles have, therefore, two sides and the included angle of the one equal to two sides and the included angle of the other, each to each; and, consequently, they are equal in all their parts (P. V.); *which was to be proved.*

*Scholium.* In triangles, equal in all their parts, the equal sides lie opposite the equal angles; and conversely.

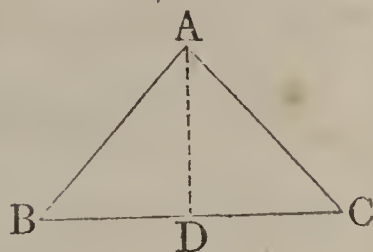
## PROPOSITION XI. THEOREM.

*In an isosceles triangle the angles opposite the equal sides are equal.*

Let  $BAC$  be an isosceles triangle, having the side  $AB$  equal to the side  $AC$ : then will the angle  $C$  be equal to the angle  $B$ .



Join the vertex  $A$  and the middle point  $D$  of the base  $BC$ . Then,  $AB$  is equal to  $AC$ , by hypothesis,  $AD$  common, and  $BD$  equal to  $DC$ , by construction: hence, the triangles  $BAD$ , and  $DAC$ , have the three sides of the one equal to those of the other, each to each; therefore, by the last Proposition, the angle  $B$  is equal to the angle  $C$ ; *which was to be proved.*



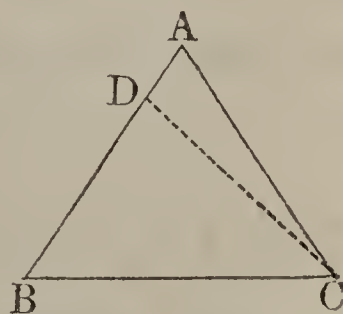
*Cor.* 1. An equilateral triangle is equiangular.

*Cor.* 2. The angle  $BAD$  is equal to  $DAC$ , and  $BDA$  to  $CDA$ : hence, the last two are right angles. Consequently, *a line drawn from the vertex of an isosceles triangle to the middle of the base, bisects the vertical angle, and is perpendicular to the base.*

#### PROPOSITION XII. THEOREM.

*If two angles of a triangle are equal, the sides opposite to them are also equal, and consequently, the triangle is isosceles.*

In the triangle  $ABC$ , let the angle  $ABC$  be equal to the angle  $ACB$ : then will  $AC$  be equal to  $AB$ , and consequently, the triangle will be isosceles.



For, if  $AB$  and  $AC$  are not equal, suppose one of them, as  $AB$ , to be the greater. On this, take  $BD$  equal to  $AC$  (Post. 3), and draw  $DC$ . Then, in the triangles  $ABC$ ,  $DBC$ , we have the side  $BD$  equal to  $AC$ , by construction, the side  $BC$  common, and the included angle  $ACB$  equal to the included angle  $DBC$ , by hypothesis: hence, the two triangles are equal

in all their parts (P. V.). But this is impossible, because a part cannot be equal to the whole (A. 8) : hence, the hypothesis that  $AB$  and  $AC$  are unequal, is false. They must, therefore, be equal ; *which was to be proved.*

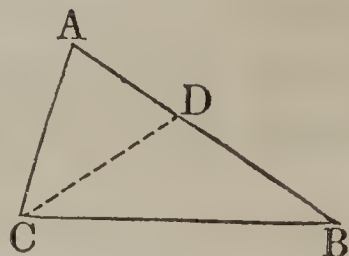
*Cor.* An equiangular triangle is equilateral.

PROPOSITION XIII. THEOREM.

*In any triangle, the greater side is opposite the greater angle ; and, conversely, the greater angle is opposite the greater side.*

In the triangle  $ABC$ , let the angle  $ACB$  be greater than the angle  $ABC$  : then will the side  $AB$  be greater than the side  $AC$ .

For, draw  $CD$ , making the angle  $BCD$  equal to the angle  $B$  (Post. 7) : then, in the triangle  $DCB$ , we have the angles  $DCB$  and  $DBC$  equal : hence, the opposite sides  $DB$  and  $DC$  are equal (P. XII.). In the triangle  $ACD$ , we have (P. VII.),



$$AD + DC > AC ;$$

or, since  $DC = DB$ , and  $AD + DB = AB$ , we have,

$$AB > AC ;$$

*which was to be proved.*

*Conversely :* Let  $AB$  be greater than  $AC$  : then will the angle  $ACB$  be greater than the angle  $ABC$ .

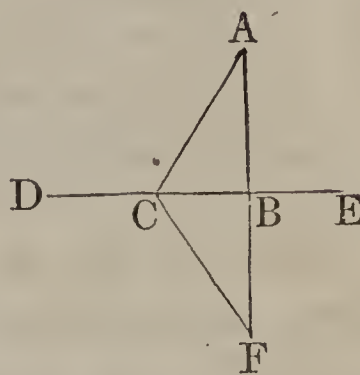
For, if  $ACB$  were less than  $ABC$ , the side  $AB$  would be less than the side  $AC$ , from what has just been proved ; if  $ACB$  were equal to  $ABC$ , the side  $AB$  would be equal to  $AC$ , by Prop. XII. ; but both conclusions are contrary

to the hypothesis: hence,  $ACB$  can neither be less than, nor equal to,  $ABC$ ; it must, therefore, be greater; *which was to be proved.*

## PROPOSITION XIV. THEOREM.

*From a given point only one perpendicular can be drawn to a given straight line.*

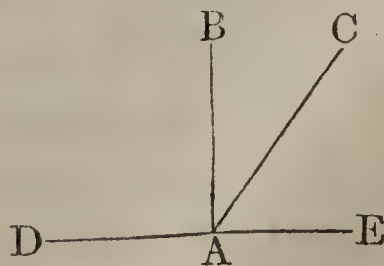
Let  $A$  be a given point, and  $AB$  a perpendicular to  $DE$ : then can no other perpendicular to  $DE$  be drawn from  $A$ .



For, suppose a second perpendicular  $AC$  to be drawn. Prolong  $AB$  till  $BF$  is equal to  $AB$ , and draw  $CF$ .

Then, the triangles  $ABC$  and  $FBC$  will have  $AB$  equal to  $BF$ , by construction,  $CB$  common, and the included angles  $ABC$  and  $FBC$  equal, because both are right angles: hence, the angles  $ACB$  and  $FCB$  are equal (P. V.) But  $ACB$  is, by a hypothesis, a right angle: hence,  $FCB$  must also be a right angle, and consequently, the line  $ACF$  must be a straight line (P. IV.). But this is impossible (A. 11). The hypothesis that two perpendiculars can be drawn is, therefore, absurd; consequently, only one such perpendicular can be drawn; *which was to be proved.*

If the given point is on the given line, the proposition is equally true. For, if from  $A$  two perpendiculars  $AB$  and  $AC$  could be drawn to  $DE$ , we should have  $BAE$  and  $CAE$  each equal to a right angle; and consequently, equal to each other; which is absurd (A. 8).



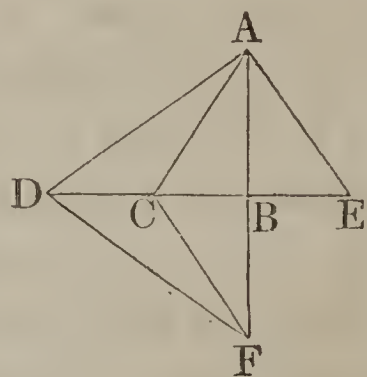


## PROPOSITION XV. THEOREM.

*If from a point without a straight line a perpendicular be let fall on the line, and oblique lines be drawn to different points of it :*

- 1°. *The perpendicular will be shorter than any oblique line:*
- 2°. *Any two oblique lines that meet the given line at points equally distant from the foot of the perpendicular, will be equal:*
- 3°. *Of two oblique lines that meet the given line at points unequally distant from the foot of the perpendicular, the one which meets it at the greater distance will be the longer.*

Let  $A$  be a given point,  $DE$  a given straight line,  $AB$  a perpendicular to  $DE$ , and  $AD$ ,  $AC$ ,  $AE$  oblique lines,  $BC$  being equal to  $BE$ , and  $BD$  greater than  $BC$ . Then will  $AB$  be less than any of the oblique lines,  $AC$  will be equal to  $AE$ , and  $AD$  greater than  $AC$ .



Prolong  $AB$  until  $BF$  is equal to  $AB$ , and draw  $FC$ ,  $FD$ .

1°. In the triangles  $ABC$ ,  $FBC$ , we have the side  $AB$  equal to  $BF$ , by construction, the side  $BC$  common, and the included angles  $ABC$  and  $FBC$  equal, because both are right angles: hence,  $FC$  is equal to  $AC$  (P. V.). But,  $AF$  is shorter than  $ACF$  (A. 12): hence,  $AB$ , the half of  $AF$ , is shorter than  $AC$ , the half of  $ACF$ ; which was to be proved.

2°. In the triangles  $ABC$  and  $ABE$ , we have the side  $BC$  equal to  $BE$ , by hypothesis, the side  $AB$  common, and the included angles  $ABC$  and  $ABE$  equal,

because both are right angles: hence,  $AC$  is equal to  $AE$ ; *which was to be proved.*

3°. It may be shown, as in the first case, that  $AD$  is equal to  $DF$ . Then, because the point  $C$  lies within the triangle  $ADF$ , the sum of the lines  $AD$  and  $DF$  will be greater than the sum of the lines  $AC$  and  $CF$  (P. VIII.): hence,  $AD$ , the half of  $ADF$ , is greater than  $AC$ , the half of  $ACF$ ; *which was to be proved.*

*Cor.* 1. The perpendicular is the shortest distance from a point to a line.

*Cor.* 2. From a given point to a given straight line, only two equal straight lines can be drawn; for, if there could be more, there would be at least two equal oblique lines on the same side of the perpendicular; which is impossible.

#### PROPOSITION XVI. THEOREM.

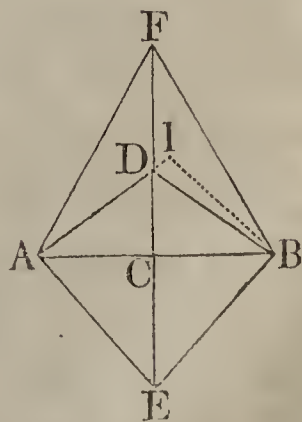
*If a perpendicular be drawn to a given straight line at its middle point:*

1°. *Any point of the perpendicular will be equally distant from the extremities of the line:*

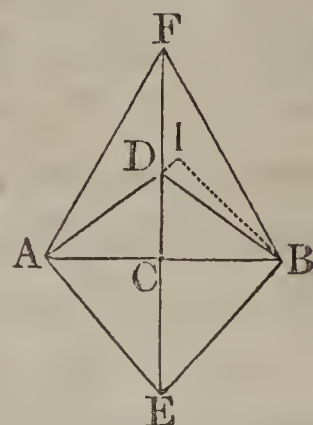
2°. *Any point, without the perpendicular, will be unequally distant from the extremities.*

Let  $AB$  be a given straight line,  $C$  its middle point, and  $EF$  the perpendicular. Then will any point of  $EF$  be equally distant from  $A$  and  $B$ ; and any point without  $EF$ , will be unequally distant from  $A$  and  $B$ .

1°. From any point of  $EF$ , as  $D$ , draw the lines  $DA$  and  $DB$ . Then will  $DA$  and  $DB$  be equal (P. XV.): hence,  $D$  is equally distant from  $A$  and  $B$ ; *which was to be proved.*



2°. From any point without  $EF$ , as  $I$ , draw  $IA$  and  $IB$ . One of these lines, as  $IA$ , will cut  $EF$  in some point  $D$ ; draw  $DB$ . Then, from what has just been shown,  $DA$  and  $DB$  will be equal; but  $IB$  is less than the sum of  $ID$  and  $DB$  (P. VII.); and because the sum of  $ID$  and  $DB$  is equal to the sum of  $ID$  and  $DA$ , or  $IA$ , we have  $IB$  less than  $IA$ : hence,  $I$  is unequally distant from  $A$  and  $B$ ; which was to be proved.

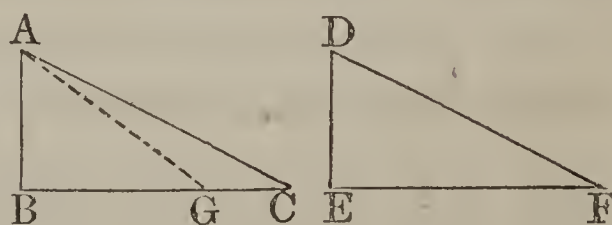


*Cor.* If a straight line  $EF$  have two of its points  $E$  and  $F$  equally distant from  $A$  and  $B$ , it will be perpendicular to the line  $AB$  at its middle point.

#### PROPOSITION XVII. THEOREM.

*If two right-angled triangles have the hypotenuse and a side of the one equal to the hypotenuse and a side of the other, each to each, the triangles will be equal in all their parts.*

Let the right-angled triangles  $ABC$  and  $DEF$  have the hypotenuse  $AC$  equal to  $DF$ , and the side  $AB$  equal to  $DE$ : then will the triangles be equal in all their parts.



If the side  $BC$  is equal to  $EF$ , the triangles will be equal, in accordance with Proposition X. Let us suppose then, that  $BC$  and  $EF$  are unequal, and that  $BC$  is the longer. On  $BC$  lay off  $BG$  equal to  $EF$ , and draw  $AG$ . The triangles  $ABG$  and  $DEF$  have  $AB$  equal to  $DE$ , by hypothesis,  $BG$  equal to  $EF$ , by construction, and



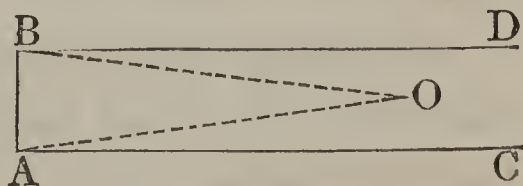
the angles  $B$  and  $E$  equal, because both are right angles; consequently,  $AG$  is equal to  $DF$  (P. V.) But,  $AC$  is equal to  $DF$ , by hypothesis: hence,  $AG$  and  $AC$  are equal, which is impossible (P. XV.). The hypothesis that  $BC$  and  $EF$  are unequal, is, therefore, absurd: hence, the triangles have all their sides equal, each to each, and are, consequently, equal in all of their parts; *which was to be proved.*

## PROPOSITION XVIII. THEOREM.

*If two straight lines are perpendicular to a third line, they will be parallel.*

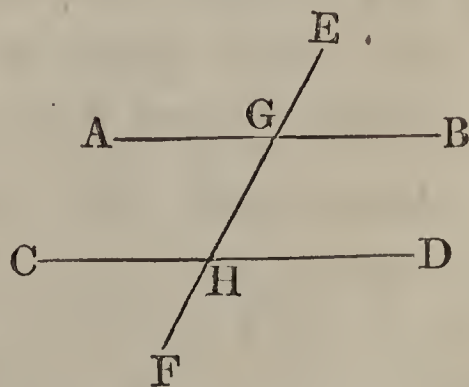
Let the two lines  $AC$ ,  $BD$ , be perpendicular to  $AB$ : then will they be parallel.

For, if they could meet in a point  $O$ , there would be two perpendiculars  $OA$ ,  $OB$ , drawn from the same point to the same straight line; which is impossible (P. XIV.): hence, the lines are parallel; *which was to be proved.*



## DEFINITIONS.

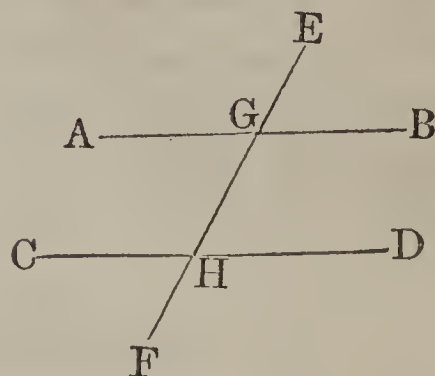
If a straight line  $EF$  intersect two other straight lines  $AB$  and  $CD$ , it is called a *secant*, with respect to them. The eight angles formed about the points of intersection have different names, with respect to each other.



1°. INTERIOR ANGLES ON THE SAME SIDE, are those that lie on the same side of the secant and *within* the other two lines. Thus,  $BGH$  and  $GHD$  are interior angles on the same side.

2°. EXTERIOR ANGLES ON THE SAME SIDE, are those that lie on the same side of the secant and *without* the other two lines. Thus,  $EGB$  and  $DHF$  are exterior angles on the same side.

3°. ALTERNATE ANGLES, are those that lie on opposite sides of the secant and *within* the other two lines, but not adjacent. Thus,  $AGH$  and  $GHD$  are alternate angles.



4°. ALTERNATE EXTERIOR ANGLES, are those that lie on opposite sides of the secant and *without* the other two lines. Thus,  $AGE$  and  $FHD$  are alternate exterior angles.

5°. OPPOSITE EXTERIOR AND INTERIOR ANGLES, are those that lie on the same side of the secant, the one *within* and the other *without* the other two lines, but not adjacent. Thus,  $EGB$  and  $GHD$  are opposite exterior and interior angles.

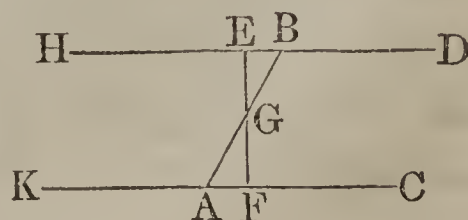
#### PROPOSITION XIX. THEOREM.

*If two straight lines, meet a third line, making the sum of the interior angles on the same side equal to two right angles, the two lines will be parallel.*

Let the lines  $KC$  and  $HD$  meet the line  $BA$ , making the sum of the angles  $BAC$  and  $ABD$  equal to two right angles: then will  $KC$  and  $HD$  be parallel.

Through  $G$ , the middle point of  $AB$ , draw  $GF$  perpendicular to  $KC$ , and prolong it to  $E$ .

The sum of the angles  $GBE$  and  $GBD$  is equal to two right



angles (P. I.); the sum of the angles  $FAG$  and  $GBD$  is equal to two right angles, by hypothesis: hence (A. 1),

$$GBE + GBD = FAG + GBD.$$

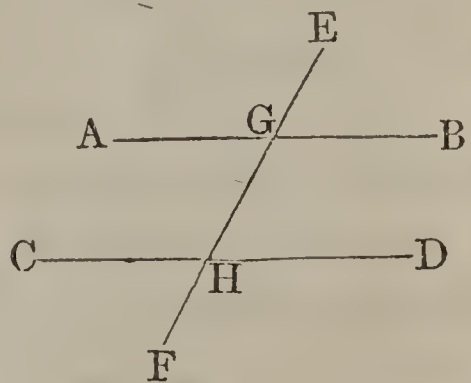
Taking from both the common part  $GBD$ , we have the angle  $GBE$  equal to the angle  $FAG$ . Again, the angles  $BGE$  and  $AGF$  are equal, because they are vertical angles (P. II.): hence, the triangles  $GEB$  and  $GFA$  have two of their angles and the included side equal, each to each; they are, therefore, equal in all their parts (P. VI.): hence, the angle  $GEB$  is equal to the angle  $GFA$ . But,  $GFA$  is a right angle, by construction;  $GEB$  must, therefore, be a right angle: hence, the lines  $KC$  and  $HD$  are both perpendicular to  $EF$ , and are, therefore, parallel (P. XVIII.); *which was to be proved.*

*Cor. 1.* If two lines are cut by a third line, making the alternate angles equal to each other, the two lines will be parallel.

Let the angle  $HGA$  be equal to  $GHD$ . Adding to both, the angle  $HGB$ , we have,

$$HGA + HGB = GHD + HGB.$$

But the first sum is equal to two right angles (P. I.): hence, the second sum is also equal to two right angles; therefore, from what has just been shown,  $AB$  and  $CD$  are parallel.



*Cor. 2.* If two lines are cut by a third, making the opposite exterior and interior angles equal, the two lines will be parallel. Let the angles  $EGB$  and  $GHD$  be equal: Now,  $EGB$  and  $AGH$  are equal, because they are vertical (P. II.); and consequently,  $AGH$  and  $GHD$  are equal: hence, from *Cor. 1*,  $AB$  and  $CD$  are parallel.

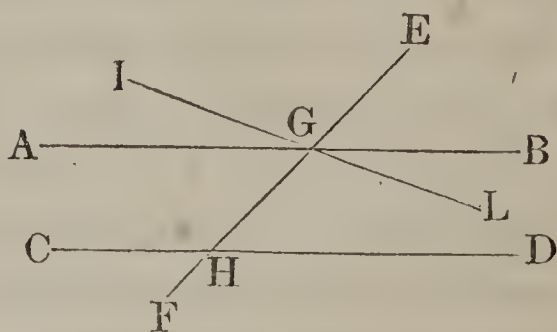


## PROPOSITION XX. THEOREM.

*If a straight line intersect two parallel straight lines, the sum of the interior angles on the same side will be equal to two right angles.*

Let the parallels  $AB$ ,  $CD$ , be cut by the secant line  $FE$ : then will the sum of  $HGB$  and  $GHD$  be equal to two right angles.

For, if the sum of  $HGB$  and  $GHD$  is not equal to two right angles, let  $IGL$  be drawn, making the sum of  $HGL$  and  $GHD$  equal to two right angles; then  $IL$  and  $CD$  will be parallel (P. XIX.); and consequently, we shall have two lines  $GB$ ,  $GL$ , drawn through the same point  $G$  and parallel to  $CD$ , which is impossible (A. 13): hence, the sum of  $HGB$  and  $GHD$ , is equal to two right angles; which was to be proved.

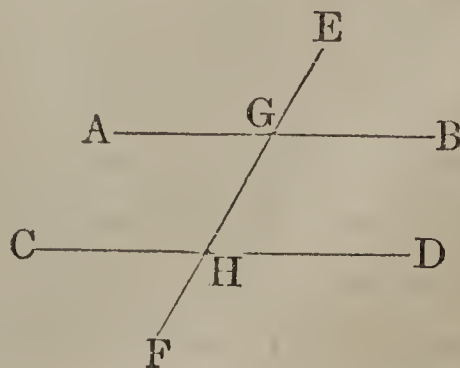


In like manner, it may be proved that the sum of  $HGA$  and  $GHC$ , is equal to two right angles.

*Cor. 1. If  $HGB$  is a right angle,  $GHD$  will be a right angle also: hence, if a line is perpendicular to one of two parallels, it is perpendicular to the other also.*

*Cor. 2. If a straight line meet two parallels, the alternate angles will be equal.*

For, if  $AB$  and  $CD$  are parallel, the sum of  $BGH$  and  $GHD$  is equal to two right angles; the sum of  $BGH$  and  $HGA$  is also equal to two right angles (P. I.): hence, these sums



are equal. Taking away the common part  $BGH$ , there remains the angle  $GHD$  equal to  $HGA$ . In like manner, it may be shown that  $BGH$  and  $GHC$  are equal.

*Cor. 3.* If a straight line meet two parallels, the opposite exterior and interior angles will be equal. The angles  $DHG$  and  $HGA$  are equal, from what has just been shown. The angles  $HGA$  and  $BGE$  are equal, because they are vertical: hence,  $DHG$  and  $BGE$  are equal. In like manner, it may be shown that  $CHG$  and  $AGE$  are equal.

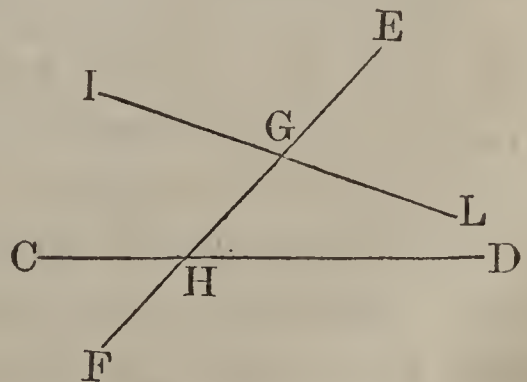
*Scholium.* Of the eight angles formed by a line cutting two parallel lines obliquely, the four acute angles are equal, and so, also, are the four obtuse angles.

#### PROPOSITION XXI. THEOREM.

*If two straight lines intersect a third line, making the sum of the interior angles on the same side less than two right angles, the two lines will meet if sufficiently produced.*

Let the two lines  $CD$ ,  $IL$ , meet the line  $EF$ , making the sum of the interior angles  $HGL$ ,  $GHD$ , less than two right angles: then will  $IL$  and  $CD$  meet if sufficiently produced.

For, if they do not meet, they must be parallel (D. 16). But, if they were parallel, the sum of the interior angles  $HGL$ ,  $GHD$ , would be equal to two right angles (P. XX.), which is contrary to the hypothesis: hence,  $IL$ ,  $CD$ , will meet if sufficiently produced; which was to be proved.



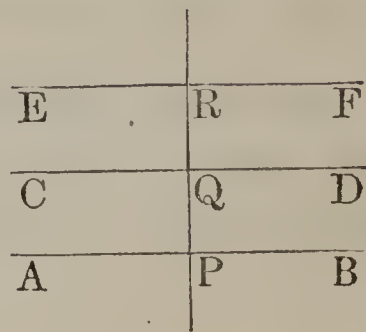
*Cor.* It is evident that  $IL$  and  $CD$ , will meet on that side of  $EF$ , on which the sum of the two angles is less than two right angles.

PROPOSITION XXII. THEOREM.

*If two straight lines are parallel to a third line, they are parallel to each other.*

Let  $AB$  and  $CD$  be respectively parallel to  $EF$ : then will they be parallel to each other.

For, draw  $PR$  perpendicular to  $EF$ ; then will it be perpendicular to  $AB$ , and also to  $CD$  (P. XX., C. 1): hence,  $AB$  and  $CD$  are perpendicular to the same straight line, and consequently, they are parallel to each other (P. XVIII.); *which was to be proved.*

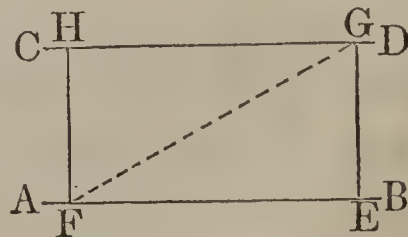


PROPOSITION XXIII. THEOREM.

*Two parallels are everywhere equally distant.*

Let  $AB$  and  $CD$  be parallel: then will they be everywhere equally distant.

From any two points of  $AB$ , as  $F$  and  $E$ , draw  $FH$  and  $EG$  perpendicular to  $CD$ ; they will also be perpendicular to  $AB$  (P. XX., C. 1), and will measure the distance between



$AB$  and  $CD$ , at the points  $F$  and  $E$ . Draw also  $FG$ . The lines  $FH$  and  $EG$  are parallel (P. XVIII.): hence, the alternate angles  $HFG$  and  $FGE$  are equal (P. XX., C. 2). The lines  $AB$  and  $CD$  are parallel, by hypothesis: hence,



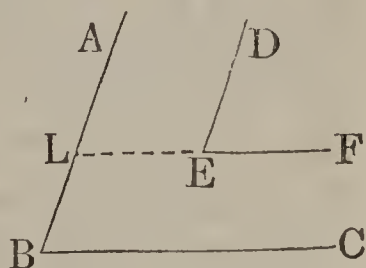
the alternate angles  $EFG$  and  $FGH$  are equal. The triangles  $FGE$  and  $FGH$  have, therefore, the angle  $HGF$  equal to  $GFE$ ,  $GFH$  equal to  $FGE$ , and the side  $FG$  common; they are, therefore, equal in all their parts (P. VI.): hence,  $FH$  is equal to  $EG$ ; and consequently,  $AB$  and  $CD$  are everywhere equally distant; *which was to be proved.*

## PROPOSITION XXIV. THEOREM.

*If two angles have their sides parallel, and lying either in the same, or in opposite directions, they will be equal.*

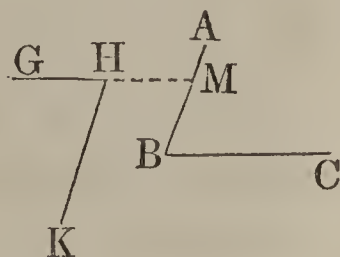
1°. Let the angles  $ABC$  and  $DEF$  have their sides parallel, and lying in the same direction: then will they be equal.

Prolong  $FE$  to  $L$ . Then, because  $DE$  and  $AL$  are parallel, the exterior angle  $DEF$  is equal to its opposite interior angle  $ALE$  (P. XX., C. 3); and because  $BC$  and  $LF$  are parallel, the exterior angle  $ALE$  is equal to its opposite interior angle  $ABC$ : hence,  $DEF$  is equal to  $ABC$ ; *which was to be proved.*



2°. Let the angles  $ABC$  and  $GHK$  have their sides parallel, and lying in opposite directions: then will they be equal.

Prolong  $GH$  to  $M$ . Then, because  $KH$  and  $BM$  are parallel, the exterior angle  $GHK$  is equal to its opposite interior angle  $HMB$ ; and because  $HM$  and  $BC$  are parallel, the angle  $HMB$  is equal to its alternate angle  $MBC$  (P. XX., C. 2): hence,  $GHK$  is equal to  $ABC$ ; *which was to be proved.*



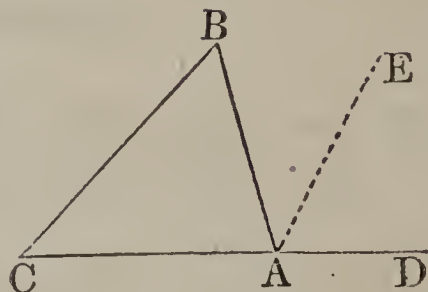
*Cor.* The opposite angles of a parallelogram are equal.

## PROPOSITION XXV. THEOREM.

*In any triangle, the sum of the three angles is equal to two right angles.*

Let  $CBA$  be any triangle: then will the sum of the angles  $C$ ,  $A$ , and  $B$ , be equal to two right angles.

For, prolong  $CA$  to  $D$ , and draw  $AE$  parallel to  $BC$ .



Then, since  $AE$  and  $CB$  are parallel, and  $CD$  cuts them, the exterior angle  $DAE$  is equal to its opposite interior angle  $C$  (P. XX., C. 3). In like manner, since  $AE$  and  $CB$  are parallel, and  $AB$  cuts them, the alternate angles  $ABC$  and  $BAE$  are equal: hence, the sum of the three angles of the triangle  $BAC$ , is equal to the sum of the angles  $CAB$ ,  $BAE$ ,  $EAD$ ; but this sum is equal to two right angles (P. I., C. 2); consequently, the sum of the three angles of the triangle, is equal to two right angles (A. 1); *which was to be proved.*

*Cor. 1.* Two angles of a triangle being given, the third will be found by subtracting their sum from two right angles.

*Cor. 2.* If two angles of one triangle are respectively equal to two angles of another, the two triangles are mutually equiangular.

*Cor. 3.* In any triangle, there can be but one right angle; for if there were two, the third angle would be zero. Nor can a triangle have more than one obtuse angle.

*Cor. 4.* In any right-angled triangle, the sum of the acute angles is equal to a right angle.

*Cor. 5.* Since every equilateral triangle is also equiangular (P. XI., C. 1), each of its angles will be equal to the third part of two right angles; so that, if the right angle is expressed by 1, each angle, of an equilateral triangle, will be expressed by  $\frac{2}{3}$ .

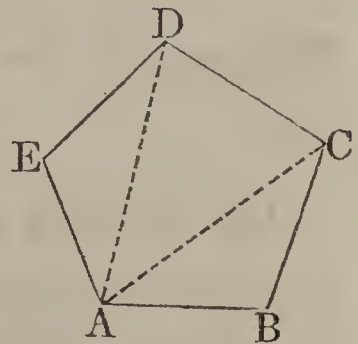
*Cor. 6.* In any triangle  $ABC$ , the exterior angle  $BAD$  is equal to the sum of the interior opposite angles  $B$  and  $C$ . For,  $AE$  being parallel to  $BC$ , the part  $BAE$  is equal to the angle  $B$ , and the other part  $DAE$ , is equal to the angle  $C$ .

PROPOSITION XXVI. THEOREM.

*The sum of the interior angles of a polygon is equal to two right angles taken as many times as the polygon has sides, less two.*

Let  $ABCDE$  be any polygon: then will the sum of its interior angles  $A$ ,  $B$ ,  $C$ ,  $D$ , and  $E$ , be equal to two right angles taken as many times as the polygon has sides, less two.

From the vertex of any angle  $A$ , draw diagonals  $AC$ ,  $AD$ . The polygon will be divided into as many triangles, less two, as it has sides, having the point  $A$  for a common vertex, and for bases, the sides of the polygon, except the two which form the



angle  $A$ . It is evident, also, that the sum of the angles of these triangles does not differ from the sum of the angles of the polygon: hence, the sum of the angles of the polygon is equal to two right angles, taken as many times as there are triangles; that is, as many times as the polygon has sides, less two; *which was to be proved.*



*Cor. 1.* The sum of the interior angles of a quadrilateral is equal to two right angles taken twice; that is, to four right angles. If the angles of a quadrilateral are equal, each will be a right angle.

*Cor. 2.* The sum of the interior angles of a pentagon is equal to two right angles taken three times; that is, to six right angles: hence, when a pentagon is equiangular, each angle is equal to the fifth part of six right angles, or to  $\frac{2}{5}$  of one right angle.

*Cor. 3.* The sum of the interior angles of a hexagon is equal to eight right angles: hence, in the equiangular hexagon, each angle is the sixth part of eight right angles, or  $\frac{4}{3}$  of one right angle.

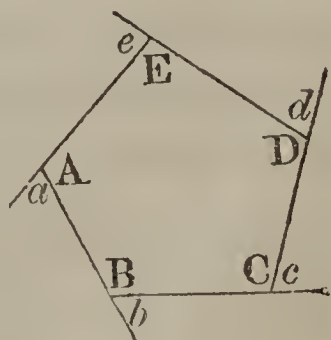
*Cor. 4.* In any equiangular polygon, any interior angle is equal to twice as many right angles as the figure has sides, less four, divided by the number of angles.

#### PROPOSITION XXVII. THEOREM.

*The sum of the exterior angles of a polygon is equal to four right angles.*

Let the sides of the polygon  $ABCDE$  be prolonged, in the same order, forming the exterior angles  $a, b, c, d, e$ ; then will the sum of these exterior angles be equal to four right angles.

For, each interior angle, together with the corresponding exterior angle, is equal to two right angles (P. I.): hence, the sum of all the interior and exterior angles is equal to two right angles taken



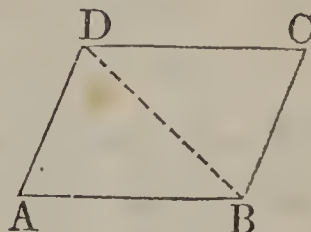
as many times as the polygon has sides. But the sum of the interior angles is equal to two right angles taken as many times as the polygon has sides, less two: hence, the sum of the exterior angles is equal to two right angles taken twice; that is, equal to four right angles; *which was to be proved.*

## PROPOSITION XXVIII. THEOREM.

*In any parallelogram, the opposite sides are equal, each to each.*

Let  $ABCD$  be a parallelogram: then will  $AB$  be equal to  $DC$ , and  $AD$  to  $BC$ .

For, draw the diagonal  $BD$ . Then, because  $AB$  and  $DC$  are parallel, the angle  $DBA$  is equal to its alternate angle  $BDC$  (P. XX., C. 2): and, because  $AD$  and  $BC$  are parallel, the angle  $BDA$  is equal to its alternate angle  $DBC$ . The triangles  $ABD$  and  $CDB$ , have, therefore, the angle  $DBA$  equal to  $CDB$ , the angle  $BDA$  equal to  $DBC$ , and the included side  $DB$  common; consequently, they are equal in all of their parts: hence,  $AB$  is equal to  $DC$ , and  $AD$  to  $BC$ ; *which was to be proved.*



*Cor. 1.* A diagonal of a parallelogram divides it into two equal triangles.

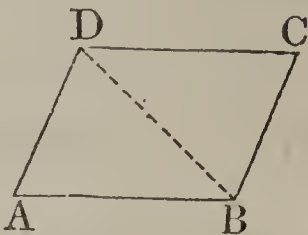
*Cor. 2.* Two parallels included between two other parallels, are equal.

*Cor. 3.* If two parallelograms have two sides and the included angle of the one, equal to two sides and the included angle of the other, each to each, they will be equal.

## PROPOSITION XXIX. THEOREM.

*If the opposite sides of a quadrilateral are equal, each to each, the figure is a parallelogram.*

In the quadrilateral  $ABCD$ , let  $AB$  be equal to  $DC$ , and  $AD$  to  $BC$ : then will it be a parallelogram.

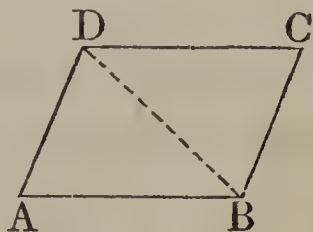


Draw the diagonal  $DB$ . Then, the triangles  $ADB$  and  $CBD$ , will have the sides of the one equal to the sides of the other, each to each; and therefore, the triangles will be equal in all of their parts: hence, the angle  $ABD$  is equal to the angle  $CDB$  (P. X., S.); and consequently,  $AB$  is parallel to  $DC$  (P. XIX., C. 1). The angle  $DBC$  is also equal to the angle  $BDA$ , and consequently,  $BC$  is parallel to  $AD$ : hence, the opposite sides are parallel, two and two; that is, the figure is a parallelogram (D. 28); *which was to be proved.*

## PROPOSITION XXX. THEOREM.

*If two sides of a quadrilateral are equal and parallel, the figure is a parallelogram.*

In the quadrilateral  $ABCD$ , let  $AB$  be equal and parallel to  $DC$ : then will the figure be a parallelogram.



Draw the diagonal  $DB$ . Then, because  $AB$  and  $DC$  are parallel, the angle  $ABD$  is equal to its alternate-angle  $CDB$ . Now, the triangles  $ABD$  and  $CDB$ , have the side  $DC$  equal to  $AB$ , by hypothesis, the side  $DB$  common, and the included angle  $ABD$  equal to  $BDC$ , from what has just



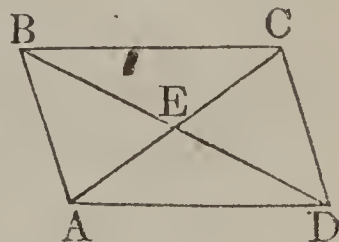
been shown : hence, the triangles are equal in all their parts (P. V.) ; and consequently, the alternate angles  $ADB$  and  $DBC$  are equal. The sides  $BC$  and  $AD$  are, therefore, parallel, and the figure is a parallelogram ; *which was to be proved.*

*Cor.* If two points be taken at equal distances from a line, and on the same side of it, the line joining them will be parallel to the given line.

PROPOSITION XXXI. THEOREM.

*The diagonals of a parallelogram divide each other into equal parts, or mutually bisect each other.*

Let  $ABCD$  be a parallelogram, and  $AC$ ,  $BD$ , its diagonals : then will  $AE$  be equal to  $EC$ , and  $BE$  to  $ED$ .



For, the triangles  $BEC$  and  $AED$ , have the angles  $EBC$  and  $ADE$  equal (P. XX., C. 2), the angles  $ECB$  and  $DAE$  equal, and the included sides  $BC$  and  $AD$  equal : hence, the triangles are equal in all of their parts (P. VI.) ; consequently,  $AE$  is equal to  $EC$ , and  $BE$  to  $ED$  ; *which was to be proved.*

*Scholium.* In a rhombus, the sides  $AB$ ,  $BC$ , being equal, the triangles  $AEB$ ,  $EBC$ , have the sides of the one equal to the corresponding sides of the other ; they are, therefore, equal : hence, the angles  $AEB$ ,  $BEC$ , are equal, and therefore, the two diagonals bisect each other at right angles.

## BOOK II.

### RATIOS AND PROPORTIONS.

#### DEFINITIONS.

1. THE RATIO of one quantity to another of the same kind, is the quotient obtained by dividing the second by the first. The first quantity is called the ANTECEDENT, and the second, the CONSEQUENT.

2. A PROPORTION is an expression of equality between two equal ratios. Thus,

$$\frac{B}{A} = \frac{D}{C},$$

expresses the fact that the ratio of  $A$  to  $B$  is equal to the ratio of  $C$  to  $D$ . In Geometry, the proportion is written thus,

$$A : B :: C : D,$$

and read,  $A$  is to  $B$ , as  $C$  is to  $D$ .

3. A CONTINUED PROPORTION is one in which several ratios are successively equal to each other ; as,

$$A : B :: C : D :: E : F :: G : H, \text{ \&c.}$$

4. There are four terms in every proportion. The first and second form the *first couplet*, and the third and fourth,

the *second couplet*. The first and fourth terms are called *extremes*; the second and third, *means*, and the fourth term, a *fourth proportional* to the other three. When the second term is equal to the third, it is said to be a *mean proportional* between the extremes. In this case, there are but three different quantities in the proportion, and the last is said to be a *third proportional to the other two*. Thus, if we have,

$$A : B :: B : C,$$

$B$  is a *mean proportional* between  $A$  and  $C$ , and  $C$  is a *third proportional* to  $A$  and  $B$ .

5. Quantities are in proportion by *alternation*, when antecedent is compared with antecedent, and consequent with consequent.

6. Quantities are in proportion by *inversion*, when antecedents are made consequents, and consequents, antecedents.

7. Quantities are in proportion by *composition*, when the sum of antecedent and consequent is compared with either antecedent or consequent.

8. Quantities are proportional by *division*, when the difference of the antecedent and consequent is compared either with antecedent or consequent.

9. Two varying quantities are *reciprocally* or *inversely* proportional, when one is increased as many times as the other is diminished. In this case, their product is a fixed quantity, as  $xy = m$ .

10. Equimultiples of two or more quantities, are the products obtained by multiplying both by the same quantity. Thus,  $mA$  and  $mB$ , are equimultiples of  $A$  and  $B$ .



## PROPOSITION I. THEOREM.

*If four quantities are in proportion, the product of the means will be equal to the product of the extremes.*

Assume the proportion,

$$A : B :: C : D; \text{ whence, } \frac{B}{A} = \frac{D}{C};$$

clearing of fractions, we have,

$$BC = AD;$$

*which was to be proved.*

*Cor.* If  $B$  is equal to  $C$ , there will be but three proportional quantities; in this case, *the square of the mean is equal to the product of the extremes.*

## PROPOSITION II. THEOREM.

*If the product of two quantities is equal to the product of two other quantities, two of them may be made the means, and the other two the extremes of a proportion.*

If we have,

$$AD = BC,$$

by changing the members of the equation, we have,

$$BC = AD;$$

dividing both members by  $AC$ , we have,

$$\frac{B}{A} = \frac{D}{C}, \text{ or } A : B :: C : D;$$

*which was to be proved.*

## PROPOSITION III. THEOREM.

*If four quantities are in proportion, they will be in proportion by alternation.*

Assume the proportion,

$$A : B :: C : D; \text{ whence, } \frac{B}{A} = \frac{D}{C}.$$

Multiplying both members by  $\frac{C}{B}$ , we have,

$$\frac{C}{A} = \frac{D}{B}; \text{ or, } A : C :: B : D;$$

*which was to be proved.*

## PROPOSITION IV. THEOREM.

*If one couplet in each of two proportions is the same, the other couplets will form a proportion.*

Assume the proportions,

$$A : B :: C : D; \text{ whence, } \frac{B}{A} = \frac{D}{C};$$

$$\text{and, } A : B :: F : G; \text{ whence, } \frac{B}{A} = \frac{G}{F}.$$

From Axiom 1, we have,

$$\frac{D}{C} = \frac{G}{F}; \text{ whence, } C : D :: F : G;$$

*which was to be proved.*

*Cor.* If the antecedents, in two proportions, are the same, the consequents will be proportional. For, the antecedents of the second couplets may be made the consequents of the first, by alternation (P. III.).

## PROPOSITION V. THEOREM.

*If four quantities are in proportion, they will be in proportion by inversion.*

Assume the proportion,

$$A : B :: C : D; \quad \text{whence, } \frac{B}{A} = \frac{D}{C}.$$

If we take the reciprocals of both members (A. 7), we have,

$$\frac{A}{B} = \frac{C}{D}; \quad \text{whence, } B : A :: D : C;$$

*which was to be proved.*

## PROPOSITION VI. THEOREM.

*If four quantities are in proportion, they will be in proportion by composition or division.*

Assume the proportion,

$$A : B :: C : D; \quad \text{whence, } \frac{B}{A} = \frac{D}{C}.$$

If we add 1 to both members, and subtract 1 from both members, we shall have,

$$\frac{B}{A} + 1 = \frac{D}{C} + 1; \quad \text{and, } \frac{B}{A} - 1 = \frac{D}{C} - 1;$$

whence, by reducing to a common denominator, we have,

$$\frac{B + A}{A} = \frac{D + C}{C}, \quad \text{and, } \frac{B - A}{A} = \frac{D - C}{C}; \quad \text{whence,}$$

$$A : B + A :: C : D + C, \quad \text{and, } A : B - A :: C : D - C$$

*which was to be proved.*



## PROPOSITION VII. THEOREM.

*Equimultiples of two quantities are proportional to the quantities themselves.*

Let  $A$  and  $B$  be any two quantities ; then  $\frac{B}{A}$  will denote their ratio.

If we multiply both terms of this fraction by  $m$ , its value will not be changed ; and we shall have,

$$\frac{mB}{mA} = \frac{B}{A} ; \quad \text{whence, } mA : mB :: A : B ;$$

*which was to be proved.*

## PROPOSITION VIII. THEOREM.

*If four quantities are in proportion, any equimultiples of the first couplet will be proportional to any equimultiples of the second couplet.*

Assume the proportion,

$$A : B :: C : D ; \quad \text{whence, } \frac{B}{A} = \frac{D}{C} .$$

If we multiply both terms of the first member by  $m$ , and both terms of the second member by  $n$ , we shall have,

$$\frac{mB}{mA} = \frac{nD}{nC} ; \quad \text{whence, } mA : mB :: nC : nD ;$$

*which was to be proved.*

## PROPOSITION IX. THEOREM.

*If two quantities be increased or diminished by like parts of each, the results will be proportional to the quantities themselves.*

We have, Prop. VII.,

$$A : B :: mA : mB.$$

If we make  $m = 1 \pm \frac{p}{q}$ , in which  $\frac{p}{q}$  is any fraction, we shall have,

$$A : B :: A \pm \frac{p}{q}A : B \pm \frac{p}{q}B;$$

*which was to be proved.*

## PROPOSITION X. THEOREM.

*If both terms of the first couplet of a proportion be increased or diminished by like parts of each; and if both terms of the second couplet be increased or diminished by any other like parts of each, the results will be in proportion.*

Since we have, Prop. VIII.,

$$mA : mB :: nC : nD;$$

if we make  $m = 1 \pm \frac{p}{q}$ , and,  $n = 1 \pm \frac{p'}{q'}$ , we shall have,

$$A \pm \frac{p}{q}A : B \pm \frac{p}{q}B :: C \pm \frac{p'}{q'}C : D \pm \frac{p'}{q'}D;$$

*which was to be proved.*

## PROPOSITION XI. THEOREM.

*In any continued proportion, the sum of the antecedents is to the sum of the consequents, as any antecedent to its corresponding consequent.*

From the definition of a continued proportion (D. 3),

$$A : B :: C : D :: E : F :: G : H, \text{ \&c.},$$

hence,

$$\frac{B}{A} = \frac{B}{A}; \quad \text{whence,} \quad BA = AB;$$

$$\frac{B}{A} = \frac{D}{C}; \quad \text{whence,} \quad BC = AD;$$

$$\frac{B}{A} = \frac{F}{E}; \quad \text{whence,} \quad BE = AF;$$

$$\frac{B}{A} = \frac{H}{G}; \quad \text{whence,} \quad BG = AH;$$

&c.,

&c.

Adding and factoring, we have,

$$B(A + C + E + G + \text{\&c.}) = A(B + D + F + H + \text{\&c.});$$

hence, from Proposition II.,

$$A + C + E + G + \text{\&c.} : B + D + F + H + \text{\&c.} :: A : B;$$

*which was to be proved.*



## PROPOSITION XII. THEOREM.

*If two proportions be multiplied together, term by term, the products will be proportional.*

Assume the two proportions,

$$A : B :: C : D; \quad \text{whence,} \quad \frac{B}{A} = \frac{D}{C};$$

$$\text{and,} \quad E : F :: G : H; \quad \text{whence,} \quad \frac{F}{E} = \frac{H}{G}.$$

Multiplying the equations, member by member, we have,

$$\frac{BF}{AE} = \frac{DH}{CG}; \quad \text{whence,} \quad AE : BF :: CG : DH;$$

*which was to be proved.*

*Cor. 1.* If the corresponding terms of two proportions are equal, each term of the resulting proportion will be the square of the corresponding term in either of the given proportions: hence, *If four quantities are proportional, their squares will be proportional.*

*Cor. 2.* If the principle of the proposition be extended to three or more proportions, and the corresponding terms of each be supposed equal, it will follow that, *like powers of proportional quantities are proportionals.*

## BOOK III.

### THE CIRCLE, AND THE MEASUREMENT OF ANGLES.

#### DEFINITIONS.

1. A CIRCLE is a plane figure, bounded by a curved line, every point of which is equally distant from a point within, called the *centre*.

The bounding line is called the *circumference*.



2. A RADIUS is a straight line drawn from the centre to any point of the circumference.

3. A DIAMETER is a straight line drawn through the centre and terminating in the circumference.

All radii of the same circle are equal. All diameters are also equal, and each is double the radius.

4. An ARC is any part of a circumference.

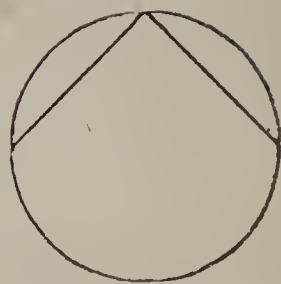
5. A CHORD is a straight line joining the extremities of an arc.

Any chord belongs to two arcs: the smaller one is meant, unless the contrary is expressed.

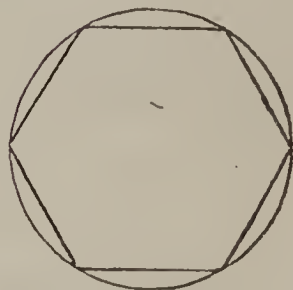
6. A SEGMENT is a part of a circle included between an arc and its chord.

7. A SECTOR is a part of a circle included within an arc and the radii drawn to its extremities.

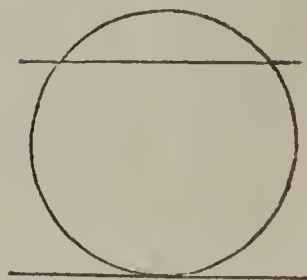
8. An **INSCRIBED ANGLE** is an angle whose vertex is in the circumference, and whose sides are chords.



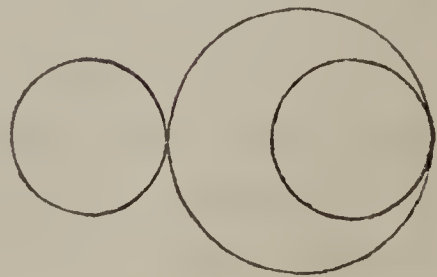
9. An **INSCRIBED POLYGON** is a polygon whose vertices are in the circumference, and whose sides are chords.



10. A **SECANT** is a straight line which cuts the circumference in two points.

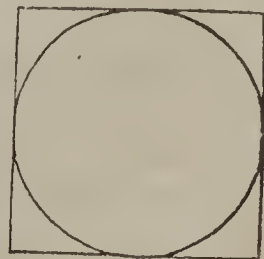


11. A **TANGENT** is a straight line which touches the circumference in one point. This point is called, the *point of contact*, or, the *point of tangency*.



12. Two circles are *tangent to each other*, when they touch each other in one point. This point is called, the *point of contact*, or the *point of tangency*.

13. A Polygon is *circumscribed about a circle*, when all of its sides are tangent to the circumference.



14. A Circle is *inscribed in a polygon*, when its circumference touches all of the sides of the polygon.

#### POSTULATE.

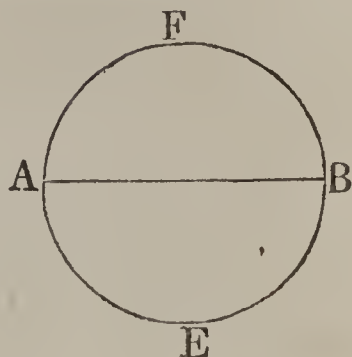
A circumference can be described from any point as a *centre*, and with any *radius*.



## PROPOSITION I. THEOREM.

*Any diameter divides the circle, and also its circumference, into two equal parts.*

Let  $AEBF$  be a circle, and  $AB$  any diameter: then will it divide the circle and its circumference into two equal parts.



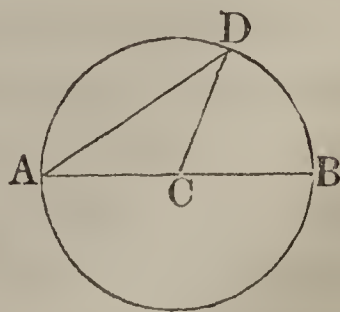
For, let  $AFB$  be applied to  $AEB$ , the diameter  $AB$  remaining common; then will they coincide; otherwise there would be some points in either one or the other of the curves unequally distant from the centre; which is impossible (D. 1): hence,  $AB$  divides the circle, and also its circumference, into two equal parts; *which was to be proved.*

## PROPOSITION II. THEOREM.

*A diameter is greater than any other chord.*

Let  $AD$  be a chord, and  $AB$  a diameter through one extremity, as  $A$ : then will  $AB$  be greater than  $AD$ .

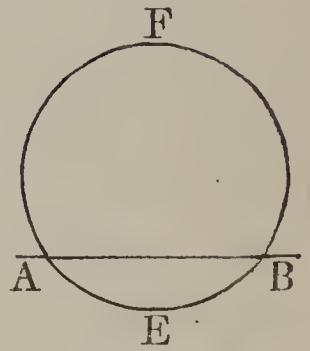
Draw the radius  $CD$ . In the triangle  $ACD$ , we have  $AD$  less than the sum of  $AC$  and  $CD$  (B. I., P. VII.). But this sum is equal to  $AB$  (D. 3): hence,  $AB$  is greater than  $AD$ ; *which was to be proved.*



## PROPOSITION III. THEOREM.

*A straight line cannot meet a circumference in more than two points.*

Let  $AEBF$  be a circumference, and  $AB$  a straight line: then  $AB$  cannot meet the circumference in more than two points.

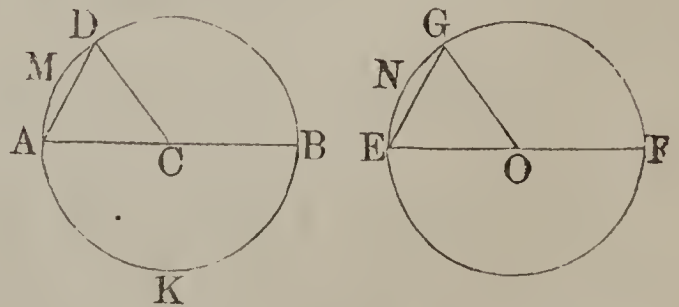


For, suppose that they could meet in three points. We should then have three equal straight lines drawn from the same point to the same straight line; which is impossible (B. I., P. XV., C. 2): hence,  $AB$  cannot meet the circumference in more than two points; *which was to be proved.*

## PROPOSITION IV. THEOREM.

*In equal circles, equal arcs are subtended by equal chords; and conversely, equal chords subtend equal arcs.*

1°. In the equal circles  $ADB$  and  $EGF$ , let the arcs  $AMD$  and  $ENG$  be equal: then will the chords  $AD$  and  $EG$  be equal.



Draw the diameters  $AB$  and  $EF$ . If the semi-circle  $ADB$  be applied to the semi-circle  $EGF$ , it will coincide with it, and the semi-circumference  $ADB$  will coincide with the semi-circumference  $EGF$ . But the part  $AMD$  is equal to the part  $ENG$ , by hypothesis: hence, the point  $D$  will fall on  $G$ ; therefore, the chord  $AD$  will coincide with

$EG$  (A. 11), and is, therefore, equal to it; *which was to be proved.*

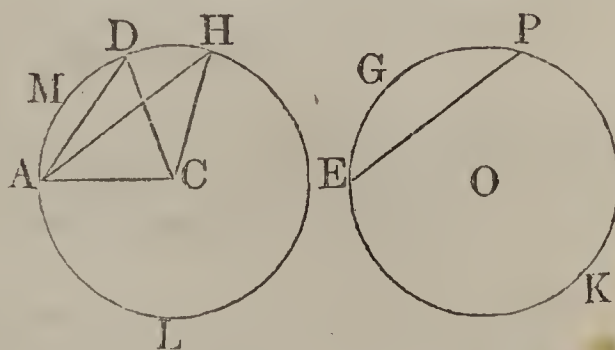
2°. Let the chords  $AD$  and  $EG$  be equal: then will the arcs  $AMD$  and  $ENG$  be equal.

Draw the radii  $CD$  and  $OG$ . The triangles  $ACD$  and  $EOG$  have all the sides of the one equal to the corresponding sides of the other; they are, therefore, equal in all their parts: hence, the angle  $ACD$  is equal to  $EOG$ . If, now, the sector  $ACD$  be placed upon the sector  $EOG$ , so that the angle  $ACD$  shall coincide with the angle  $EOG$ , the sectors will coincide throughout; and, consequently, the arcs  $AMD$  and  $ENG$  will coincide: hence, they will be equal; *which was to be proved.*

#### PROPOSITION V. THEOREM.

*In equal circles, a greater arc is subtended by a greater chord; and conversely, a greater chord subtends a greater arc.*

1°. In the equal circles  $ADL$  and  $EGK$ , let the arc  $EGP$  be greater than the arc  $AMD$ : then will the chord  $EP$  be greater than the chord  $AD$ .



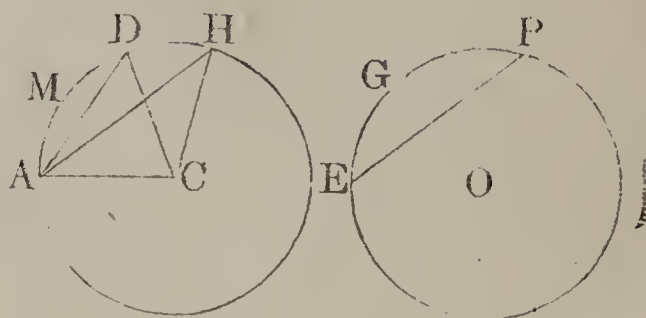
For, place the circle  $EGK$  upon  $AHL$ , so that the centre  $O$  shall fall upon the centre  $C$ , and the point  $E$  upon  $A$ ; then, because the arc  $EGP$  is greater than  $AMD$ , the point  $P$  will fall at some point  $H$ , beyond  $D$ , and the chord  $EP$  will take the position  $AH$ .

Draw the radii  $CA$ ,  $CD$ , and  $CH$ . Now, the sides  $AC$ ,  $CH$ , of the triangle  $ACH$ , are equal to the sides  $AC$ ,  $CD$ , of the triangle  $ACD$ , and the angle  $ACH$  is



greater than  $ACD$ : hence, the side  $AH$ , or its equal  $EP$ , is greater than the side  $AD$  (B. I., P. IX.); *which was to be proved.*

2°. Let the chord  $EP$ , or its equal  $AH$ , be greater than  $AD$ : then will the arc  $EGP$ , or its equal  $ADH$ , be greater than  $AMD$ .



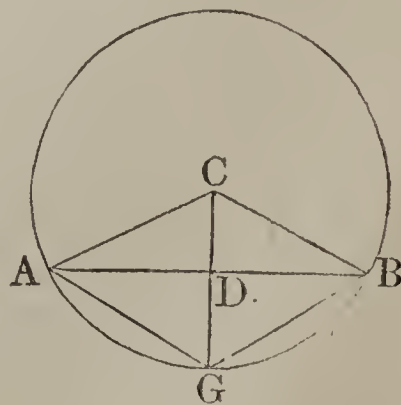
For, if  $ADH$  were equal to  $AMD$ , the chord  $AH$  would be equal to the chord  $AD$  (P. IV.); which is contrary to the hypothesis. And, if the arc  $ADH$  were less than  $AMD$ , the chord  $AH$  would be less than  $AD$ ; which is also contrary to the hypothesis. Then, since the arc  $ADH$ , subtended by the greater chord, can neither be equal to, nor less than  $AMD$ , it must be greater than  $AMD$ ; *which was to be proved.*

#### PROPOSITION VI. THEOREM.

*The radius which is perpendicular to a chord, bisects that chord, and also the arc subtended by it.*

Let  $CG$  be the radius which is perpendicular to the chord  $AB$ : then will this radius bisect the chord  $AB$ , and also the arc  $AGB$ .

For, draw the radii  $CA$  and  $CB$ . Then, the right-angled triangles  $CDA$  and  $CDB$  will have the hypotenuse  $CA$  equal to  $CB$ , and the side  $CD$  common; the triangles are, therefore, equal in all their parts: hence,  $AD$  is equal to  $DB$ . Again, because  $CG$



is perpendicular to  $AB$ , at its middle point, the chords  $GA$  and  $GB$  are equal (B. I., P. XVI.) ; and consequently, the arcs  $GA$  and  $GB$  are also equal (P. IV.) : hence,  $CG$  bisects the chord  $AB$ , and also the arc  $AGB$  ; *which was to be proved.*

*Cor.* A straight line, perpendicular to a chord, at its middle point, passes through the centre of the circle.

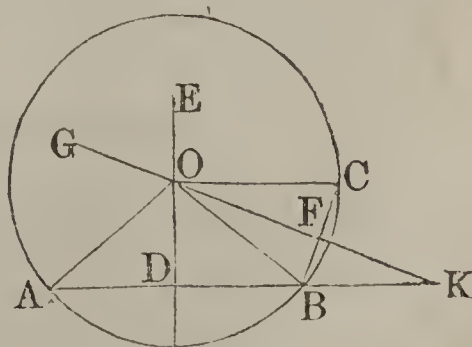
*Scholium.* The centre  $C$ , the middle point  $D$  of the chord  $AB$ , and the middle point  $G$  of the subtended arc, are points of the radius perpendicular to the chord. But two points determine the position of a straight line (A. 11) : hence, any straight line which passes through two of these points, will pass through the third, and be perpendicular to the chord.

#### PROPOSITION VII. THEOREM.

*Through any three points, not in the same straight line, one circumference may be made to pass, and but one.*

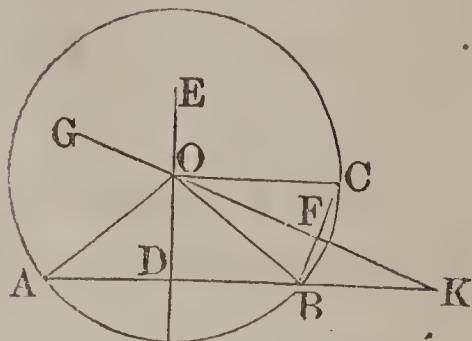
Let  $A$ ,  $B$ , and  $C$ , be any three points, not in a straight line : then may one circumference be made to pass through them, and but one.

Join the points by the lines  $AB$ ,  $BC$ , and bisect these lines by perpendiculars  $DE$  and  $FG$  : then will these perpendiculars meet in some point  $O$ . For, if they do not meet, they are parallel ; and if they are parallel, the line  $ABK$ , which is perpendicular to  $DE$ , is also perpendicular to  $KG$  (B. I., P. XX., C. 1) ; consequently, there are two lines  $BK$  and  $BF$ , drawn through the same



point  $B$ , and perpendicular to the same line  $KG$ ; which is impossible: hence,  $DE$  and  $FG$  meet in some point  $O$ .

Now,  $O$  is on a perpendicular to  $AB$  at its middle point, it is, therefore, equally distant from  $A$  and  $B$  (B. I., P. XVI.). For a like reason,  $O$  is equally distant from  $B$  and  $C$ . If, therefore, a circumference be described from  $O$  as a centre, with a radius equal to  $OA$ , it will pass through  $A$ ,  $B$ , and  $C$ .



Again,  $O$  is the only point which is equally distant from  $A$ ,  $B$ , and  $C$ : for,  $DE$  contains all of the points which are equally distant from  $A$  and  $B$ ; and  $FG$  all of the points which are equally distant from  $B$  and  $C$ ; and consequently, their point of intersection  $O$ , is the only point that is equally distant from  $A$ ,  $B$ , and  $C$ : hence, one circumference may be made to pass through these points, and but one; *which was to be proved*.

*Cor.* Two circumferences cannot intersect in more than two points; for, if they could intersect in three points, there would be two circumferences passing through the same three points; which is impossible.

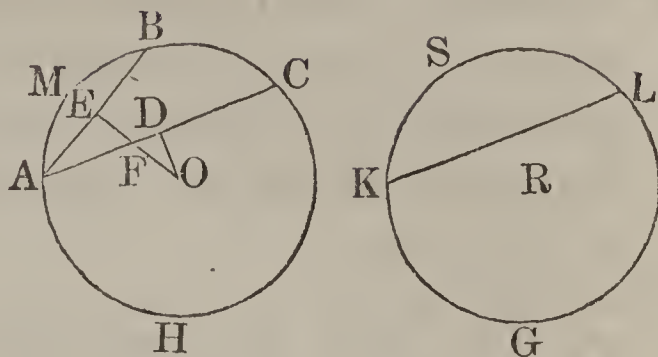
#### PROPOSITION VIII. THEOREM.

*In equal circles, equal chords are equally distant from the centres; and of two unequal chords, the less is at the greater distance from the centre.*

1°. In the equal circles  $ACH$  and  $KLG$ , let the chords  $AC$  and  $KL$  be equal: then will they be equally distant from the centres.



For, let the circle  $KLG$  be placed upon  $ACH$ , so that the centre  $R$  shall fall upon the centre  $O$ , and the point  $K$  upon the point  $A$ : then will the chord  $KL$  coincide with  $AC$  (P. IV.); and consequently, they will be equally distant from the centre; *which was to be proved.*



2°. Let  $AB$  be less than  $KL$ : then will it be at a greater distance from the centre.

For, place the circle  $KLG$  upon  $ACH$ , so that  $R$  shall fall upon  $O$ , and  $K$  upon  $A$ . Then, because the chord  $KL$  is greater than  $AB$ , the arc  $KSL$  is greater than  $AMB$ ; and consequently, the point  $L$  will fall at a point  $C$ , beyond  $B$ , and the chord  $KL$  will take the direction  $AC$ .

Draw  $OD$  and  $OE$ , respectively perpendicular to  $AC$  and  $AB$ ; then will  $OE$  be greater than  $OF$  (A. 8), and  $OF$  than  $OD$  (B. I., P. XV.): hence,  $OE$  is greater than  $OD$ . But,  $OE$  and  $OD$  are the distances of the two chords from the centre (B. I., P. XV., C. 1): hence, the less chord is at the greater distance from the centre; *which was to be proved.*

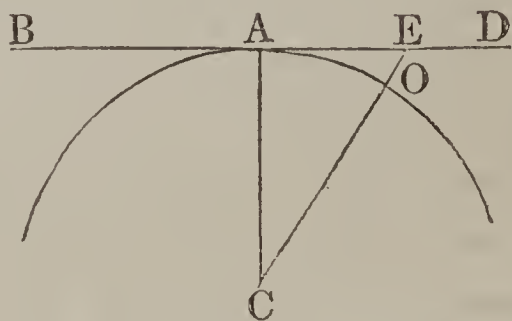
*Scholium.* All the propositions relating to chords and arcs of equal circles, are also true for chords and arcs of one and the same circle. For, any circle may be regarded as made up of two equal circles, so placed, that they coincide in all their parts.

## PROPOSITION IX. THEOREM.

*If a straight line is perpendicular to a radius at its extremity, it will be tangent to the circle at that point; conversely, if a straight line is tangent to a circle at any point, it will be perpendicular to the radius drawn to that point.*

1°. Let  $BD$  be perpendicular to the radius  $CA$ , at  $A$ : then will it be tangent to the circle at  $A$ .

For, take any other point of  $BD$ , as  $E$ , and draw  $CE$ : then will  $CE$  be greater than  $CA$  (B. I., P. XV.); and consequently, the point  $E$  will lie without the circle: hence,  $BD$  touches the circumference at the



point  $A$ ; it is, therefore, tangent to it at that point (D. 11); *which was to be proved.*

2°. Let  $BD$  be tangent to the circle at  $A$ : then will it be perpendicular to  $CA$ .

For, let  $E$  be any point of the tangent, except the point of contact, and draw  $CE$ . Then, because  $BD$  is a tangent,  $E$  lies without the circle; and consequently,  $CE$  is greater than  $CA$ : hence,  $CA$  is shorter than any other line that can be drawn from  $C$  to  $BD$ ; it is, therefore, perpendicular to  $BD$  (B. I., P. XV., C. 1); *which was to be proved.*

*Cor.* At a given point of a circumference, only one tangent can be drawn. For, if two tangents could be drawn, they would both be perpendicular to the same radius at the same point; which is impossible (B. I., P. XIV.).

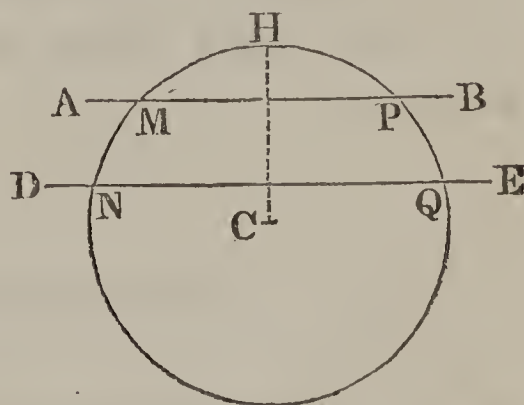
## PROPOSITION X. THEOREM.

*Two parallels intercept equal arcs of a circumference.*

There may be three cases: both parallels may be secants; one may be a secant and the other a tangent; or, both may be tangents.

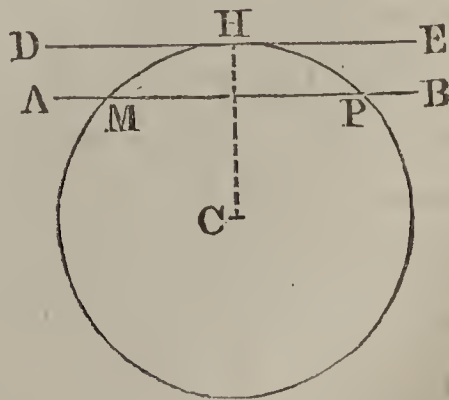
1°. Let the secants  $AB$  and  $DE$  be parallel: then will the intercepted arcs  $MN$  and  $PQ$  be equal.

For, draw the radius  $CH$  perpendicular to the chord  $MP$ ; it will also be perpendicular to  $NQ$  (B. I., P. XX., C. 1), and  $H$  will be at the middle point of the arc  $MHP$ , and also of the arc  $NHQ$ : hence,  $MN$ , which is the difference of  $HN$  and  $HM$ , is equal to  $PQ$ , which is the difference of  $HQ$  and  $HP$  (A. 3); *which was to be proved.*



2°. Let the secant  $AB$  and tangent  $DE$ , be parallel: then will the intercepted arcs  $MH$  and  $PH$  be equal.

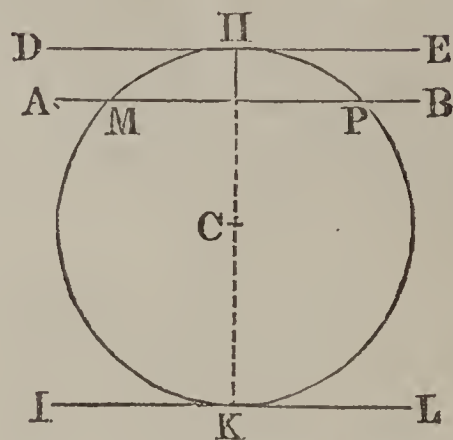
For, draw the radius  $CH$  to the point of contact  $H$ ; it will be perpendicular to  $DE$  (P. IX.), and also to its parallel  $MP$ . But, because  $CH$  is perpendicular to  $MP$ ,  $H$  is the middle point of the arc  $MHP$  (P. VI.): hence,  $MH$  and  $PH$  are equal; *which was to be proved.*





3°. Let the tangents  $DE$  and  $IL$  be parallel, and let  $H$  and  $K$  be their points of contact: then will the intercepted arcs  $HMK$  and  $HPK$  be equal.

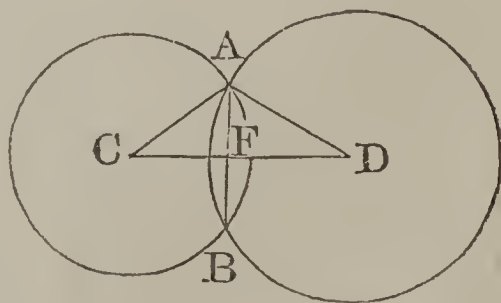
For, draw the secant  $AB$  parallel to  $DE$ ; then, from what has just been shown, we shall have  $HM$  equal to  $HP$ , and  $MK$  equal to  $PK$ : hence,  $HMK$ , which is the sum of  $HM$  and  $MK$ , is equal to  $HPK$ , which is the sum of  $HP$  and  $PK$ ; which was to be proved.



#### PROPOSITION XI. THEOREM.

*If two circumferences intersect each other, the points of intersection will be in a perpendicular to the line joining their centres, and at equal distances from it.*

Let the circumferences, whose centres are  $C$  and  $D$ , intersect at the points  $A$  and  $B$ : then will  $CD$  be perpendicular to  $AB$ , and  $AF$  will be equal to  $BF$ .

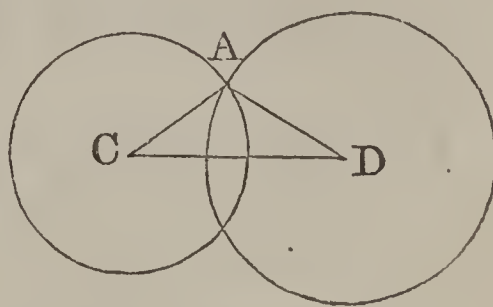


For, the points  $A$  and  $B$ , being on the circumference whose centre is  $C$ , are equally distant from  $C$ ; and being on the circumference whose centre is  $D$ , they are equally distant from  $D$ : hence,  $CD$  is perpendicular to  $AB$  at its middle point (B. I., P. XVI., C.); which was to be proved.

## PROPOSITION XII. THEOREM.

*If two circumferences intersect each other, the distance between their centres will be less than the sum, and greater than the difference, of their radii.*

Let the circumferences, whose centres are  $C$  and  $D$ , intersect at  $A$ : then will  $CD$  be less than the sum, and greater than the difference of the radii of the two circles.



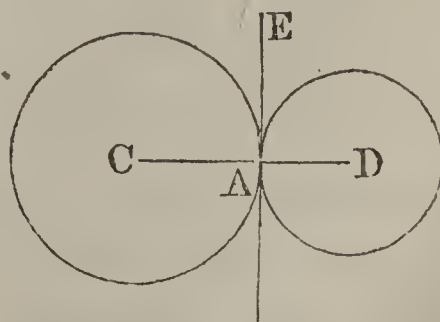
For, draw  $AC$  and  $AD$ , forming the triangle  $ACD$ . Then will  $CD$  be less than the sum of  $AC$  and  $AD$ , and greater than their difference (B. I., P. VII.) ; which was to be proved.

## PROPOSITION XIII. THEOREM.

*If the distance between the centres of two circles is equal to the sum of their radii, they will be tangent externally.*

Let  $C$  and  $D$  be the centres of two circles, and let the distance between the centres be equal to the sum of the radii: then will the circles be tangent externally.

For, they will have a point  $A$ , on the line  $CD$ , common, and they will have no other point in common; for, if they had two points in common, the distance between their centres would be less than the sum of



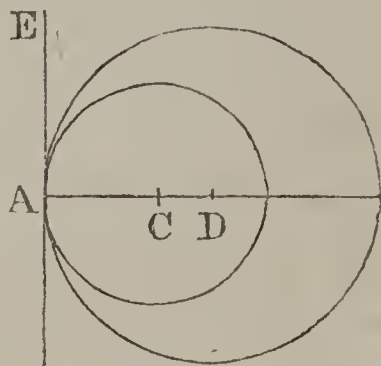
their radii; which is contrary to the hypothesis: hence, they are tangent externally; which was to be proved.

## PROPOSITION XIV. THEOREM.

*If the distance between the centres of two circles is equal to the difference of their radii, one will be tangent to the other internally.*

Let  $C$  and  $D$  be the centres of two circles, and let the distance between these centres be equal to the difference of the radii: then will the one be tangent to the other internally.

For, they will have a point  $A$ , on  $DC$ , common, and they will have no other point in common. For, if they had two points in common, the distance between their centres would be greater than the difference of their radii; which is contrary to the hypothesis: hence, one touches the other internally; *which was to be proved.*



*Cor. 1.* If two circles are tangent, either externally or internally, the point of contact will be on the straight line drawn through their centres.

*Cor. 2.* All circles whose centres are on the same straight line, and which pass through a common point of that line, are tangent to each other at that point. And if a straight line be drawn tangent to one of the circles at their common point, it will be tangent to them all at that point.

*Scholium.* From the preceding propositions, we infer that two circles may have any one of six positions with respect to each other, depending upon the distance between their centres:

1°. When the distance between their centres is greater



than the sum of their radii, *they are external, one to the other*:

2°. When this distance is equal to the sum of the radii, *they are tangent, externally*:

3°. When this distance is less than the sum, and greater than the difference of the radii, *they intersect each other*:

4°. When this distance is equal to the difference of their radii, *one is tangent to the other, internally*:

5°. When this distance is less than the difference of the radii, *one is wholly within the other*:

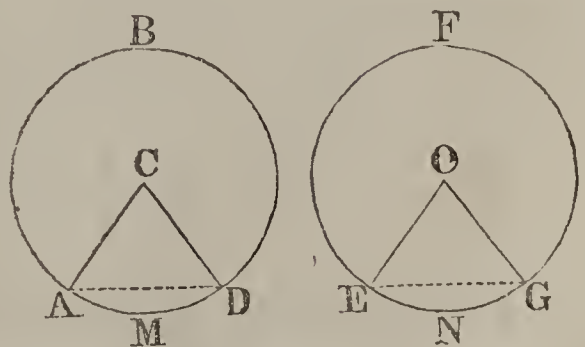
6°. When this distance is equal to zero, *they have a common centre; or, they are concentric*.

PROPOSITION XV. THEOREM.

*In equal circles, radii making equal angles at the centre, intercept equal arcs of the circumference; conversely, radii which intercept equal arcs, make equal angles at the centre.*

1°. In the equal circles  $ADB$  and  $EGF$ , let the angles  $ACD$  and  $EOG$  be equal: then will the arcs  $AMD$  and  $ENG$  be equal.

For, draw the chords  $AD$  and  $EG$ ; then will the triangles  $ACD$  and  $EOG$  have two sides and their included angle, in the one, equal to two sides and their included

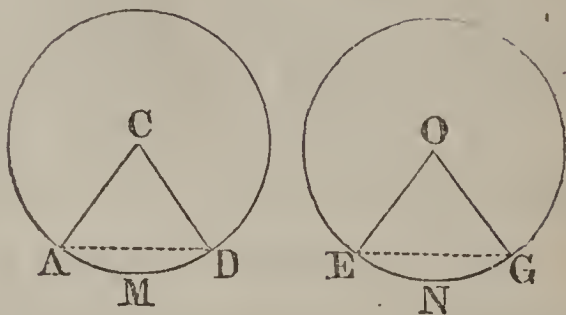


angle, in the other, each to each. They are, therefore, equal in all their parts; consequently,  $AD$  is equal to  $EG$ . But, if the chords  $AD$  and  $EG$  are equal, the arcs  $AMD$  and  $ENG$  are also equal (P. IV.); *which was to be proved.*

2°. Let the arcs  $AMD$  and  $ENG$  be equal: then will the angles  $ACD$  and  $EOG$  be equal.

For, if the arcs  $AMD$  and  $ENG$  are equal, the chords  $AD$  and  $EG$  are equal (P. IV.); consequently, the triangles  $ACD$  and  $EOG$  have their sides equal, each to each; they are, therefore,

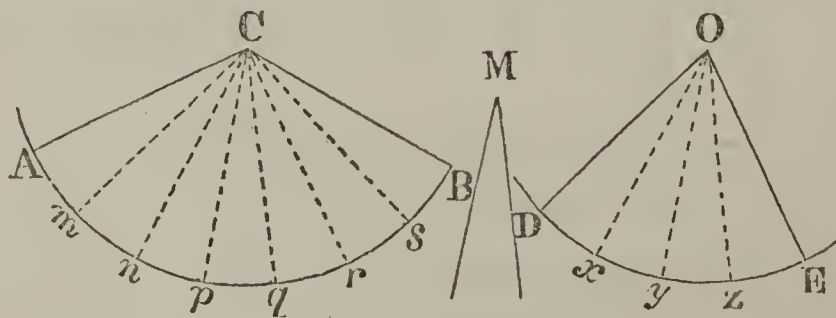
equal in all their parts: hence, the angle  $ACD$  is equal to the angle  $EOG$ ; which was to be proved.



#### PROPOSITION XVI. THEOREM.

*In equal circles, commensurable angles at the centre are proportional to their intercepted arcs.*

In the equal circles, whose centres are  $C$  and  $O$ , let the angles  $ACB$  and  $DOE$  be commensurable; that is, let them have a common unit: then will they be proportional to the intercepted arcs  $AB$  and  $DE$ .



Let the angle  $M$  be a common unit; and suppose, for example, that this unit is contained 7 times in the angle  $ACB$ , and 4 times in the angle  $DOE$ . Then, suppose  $ACB$  be divided into 7 angles, by the radii  $Cm$ ,  $Cn$ ,  $Cp$ , &c.; and  $DOE$  into 4 angles, by the radii  $Ox$ ,  $Oy$ , and  $Oz$ , each equal to the unit  $M$ .

From the last proposition, the arcs  $Am$ ,  $mn$ , &c.,  $Dx$ ,  $xy$ , &c., are equal to each other; and because there are 7 of these arcs in  $AB$ , and 4 in  $DE$ , we shall have,

$$\text{arc } AB : \text{arc } DE :: 7 : 4.$$

But, by hypothesis, we have,

$$\text{angle } ACB : \text{angle } DOE :: 7 : 4;$$

hence, from (B. II., P. IV.), we have,

$$\text{angle } ACB : \text{angle } DOE :: \text{arc } AB : \text{arc } DE.$$

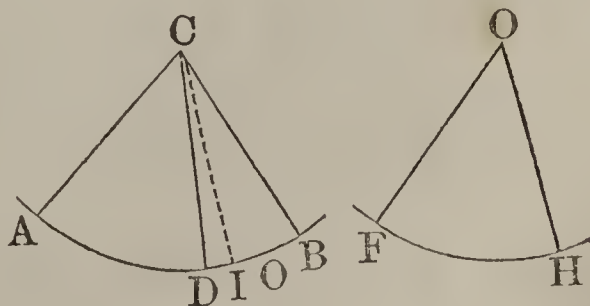
If any other numbers than 7 and 4 had been used, the same proportion would have been found; *which was to be proved.*

*Cor.* If the intercepted arcs are commensurable, they will be proportional to the corresponding angles at the centre, as may be shown by changing the order of the complements in the above proportion.

PROPOSITION XVII. THEOREM.

*In equal circles, incommensurable angles are proportional to their intercepted arcs.*

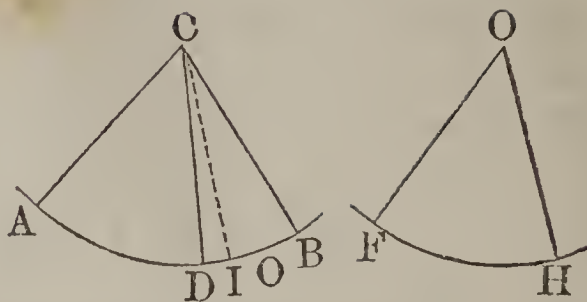
In the equal circles, whose centres are  $C$  and  $O$ , let  $ACB$  and  $FOH$  be incommensurable: then will they be proportional to the arcs  $AB$  and  $FH$ .



For, let the less angle  $FOH$ , be placed upon the greater angle  $ACB$ , so that it shall take the position  $ACD$ .



Then, if the proposition is not true, let us suppose that the angle  $ACB$  is to the angle  $FOH$ , or its equal  $ACD$ , as the arc  $AB$  is to an arc  $AO$ , greater than  $FH$ , or its equal  $AD$ ; whence,



$$\text{angle } ACB : \text{angle } ACD :: \text{arc } AB : \text{arc } AO.$$

Conceive the arc  $AB$  to be divided into equal parts, each less than  $DO$ : there will be at least one point of division between  $D$  and  $O$ ; let  $I$  be that point; and draw  $CI$ . Then the arcs  $AB$ ,  $AI$ , will be commensurable, and we shall have (P. XVI.),

$$\text{angle } ACB : \text{angle } ACI :: \text{arc } AB : \text{arc } AI.$$

Comparing the two proportions, we see that the antecedents are the same in both: hence, the consequents are proportional (B. II., P. IV., C.); hence,

$$\text{angle } ACD : \text{angle } ACI :: \text{arc } AO : \text{arc } AI.$$

But,  $AO$  is greater than  $AI$ : hence, if this proportion is true, the angle  $ACD$  must be greater than the angle  $ACI$ . On the contrary, it is less: hence, the fourth term of the proportion cannot be greater than  $AD$ .

In a similar manner, it may be shown that the fourth term cannot be less than  $AD$ : hence, it must be equal to  $AD$ ; therefore, we have,

$$\text{angle } ACB : \text{angle } ACD :: \text{arc } AB : \text{arc } AD;$$

*which was to be proved.*

*Cor. 1.* The intercepted arcs are proportional to the cor-

responding angles at the centre, as may be shown by changing the order of the couplets in the preceding proportion.

*Cor. 2.* In equal circles, angles at the centre are proportional to their intercepted arcs; and the reverse, whether they are commensurable or incommensurable.

*Cor. 3.* In equal circles, sectors are proportional to their angles, and also to their arcs.

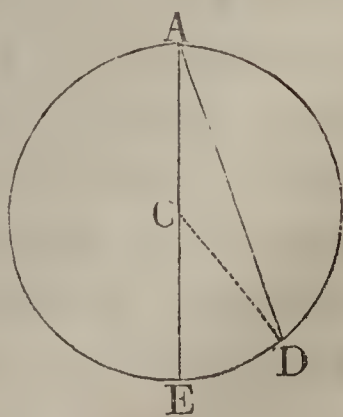
*Scholium.* Since the intercepted arcs are proportional to the corresponding angles at the centre, the arcs may be taken as the measures of the angles. That is, if a circumference be described from the vertex of any angle, as a centre, and with a fixed radius, the arc intercepted between the sides of the angle may be taken as the measure of the angle. In Geometry, the right angle which is measured by a quarter of a circumference, or a *quadrant*, is taken as a unit. If, therefore, any angle be measured by one-half or two-thirds of a quadrant, it will be equal to one-half or two-thirds of a right angle.

#### PROPOSITION XVIII. THEOREM.

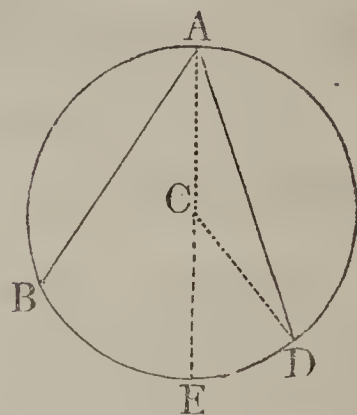
*An inscribed angle is measured by half of the arc included between its sides.*

There may be three cases: the centre of the circle may lie on one of the sides of the angle; it may lie within the angle; or, it may lie without the angle.

1°. Let  $EAD$  be an inscribed angle, one of whose sides  $AE$  passes through the centre: then will it be measured by half of the arc  $DE$ .



For, draw the radius  $CD$ . The external angle  $DCE$ , of the triangle  $DCA$ , is equal to the sum of the opposite interior angles  $CAD$  and  $CDA$  (B. I., P. XXV., C. 6). But, the triangle  $DCA$  being isosceles, the angles  $D$  and  $A$  are equal; therefore, the angle  $DCE$  is double the angle  $DAE$ . Because  $DCE$  is at the centre, it is measured by the arc  $DE$  (P. XVII., S.): hence, the, angle  $DAE$  is measured by half of the arc  $DE$ ; *which was to be proved.*

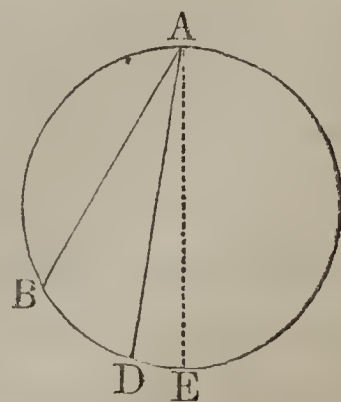


2°. Let  $DAB$  be an inscribed angle, and let the centre lie within it: then will the angle be measured by half of the arc  $BED$ .

For, draw the diameter  $AE$ . Then, from what has just been proved, the angle  $DAE$  is measured by half of  $DE$ , and the angle  $EAB$  by half of  $EB$ : hence,  $BAD$ , which is the sum of  $EAB$  and  $DAE$ , is measured by half of the sum of  $DE$  and  $EB$ , or by half of  $BED$ ; *which was to be proved.*

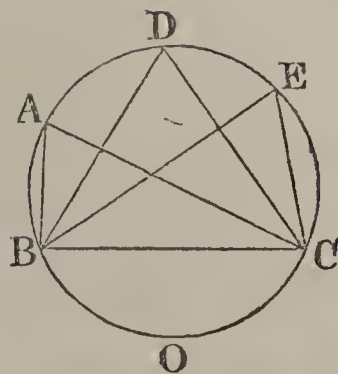
3°. Let  $BAD$  be an inscribed angle, and let the centre lie without it: then will it be measured by half of the arc  $BD$ .

For, draw the diameter  $AE$ . Then, from what precedes, the angle  $DAE$  is measured by half of  $DE$ , and the angle  $BAE$  by half of  $BE$ : hence,  $BAD$ , which is the difference of  $BAE$  and  $DAE$ , is measured by half of the difference of  $BE$  and  $DE$ , or by half of the arc  $BD$ ; *which was to be proved.*

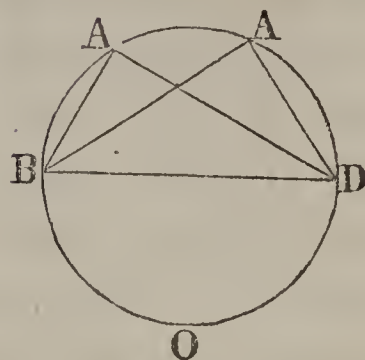




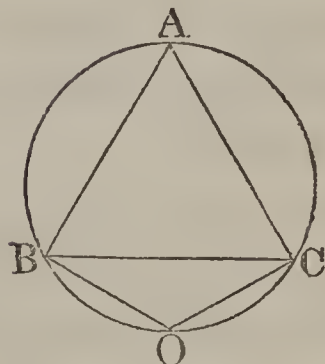
*Cor. 1.* All the angles  $BAC$ ,  $BDC$ ,  $BEC$ , inscribed in the same segment, are equal; because they are each measured by half of the same arc  $BOC$ .



*Cor. 2.* Any angle  $BAD$ , inscribed in a semi-circle, is a right angle; because it is measured by half the semi-circumference  $BOD$ , or by a quadrant (P. XVII., S.).

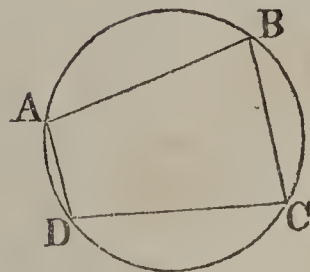


*Cor. 3.* Any angle  $BAC$ , inscribed in a segment greater than a semi-circle, is acute; for it is measured by half the arc  $BOC$ , less than a semi-circumference.



Any angle  $BOC$ , inscribed in a segment less than a semi-circle, is obtuse; for it is measured by half the arc  $BAC$ , greater than a semi-circumference.

*Cor. 4.* The opposite angles  $A$  and  $C$ , of an inscribed quadrilateral  $ABCD$ , are together equal to two right angles; for the angle  $DAB$  is measured by half the arc  $DCB$ , the angle  $DCB$  by half the arc  $DAB$ : hence, the two angles, taken together, are measured by half the circumference: hence, their sum is equal to two right angles.

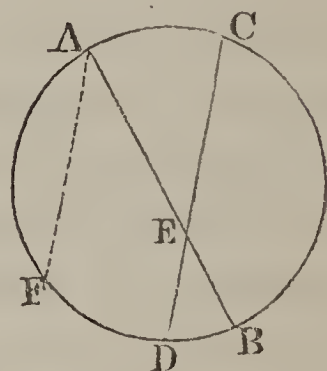


## PROPOSITION XIX. THEOREM.

*Any angle formed by two chords, which intersect, is measured by half the sum of the included arcs.*

Let  $DEB$  be an angle formed by the intersection of the chords  $AB$  and  $CD$ : then will it be measured by half the sum of the arcs  $AC$  and  $DB$ .

For, draw  $AF$  parallel to  $DC$ : then, the arc  $DF$  will be equal to  $AC$  (P. X.), and the angle  $FAB$  equal to the angle  $DEB$  (B. I., P. XX., C. 3). But the angle  $FAB$  is measured by half the arc  $FDB$  (P. XVIII.); therefore,  $DEB$  is measured by half of  $FDB$ ; that is, by half the sum of  $FD$  and  $DB$ , or by half the sum of  $AC$  and  $DB$ ; which was to be proved.

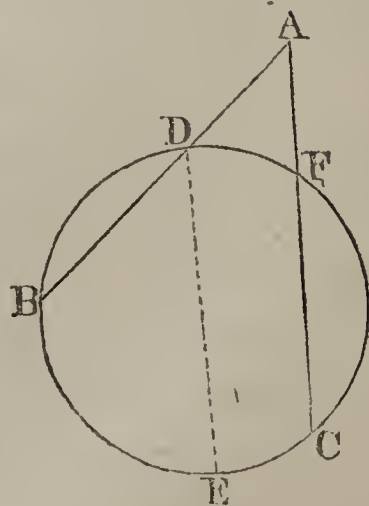


## PROPOSITION XX. THEOREM.

*The angle formed by two secants, is measured by half the difference of the included arcs.*

Let  $AB$ ,  $AC$ , be two secants: then will the angle  $BAC$  be measured by half the difference of the arcs  $BC$  and  $DF$ .

Draw  $DE$  parallel to  $AC$ : the arc  $EC$  will be equal to  $DF$  (P. X.), and the angle  $BDE$  equal to the angle  $BAC$  (B. I., P. XX., C. 3.). But  $BDE$  is measured by half the arc  $BE$  (P. XVIII.): hence,  $BAC$  is also measured by half the arc  $BE$ ; that is, by half the difference of  $BC$  and  $EC$ , or by half the difference of  $BC$  and  $DF$ ; which was to be proved.

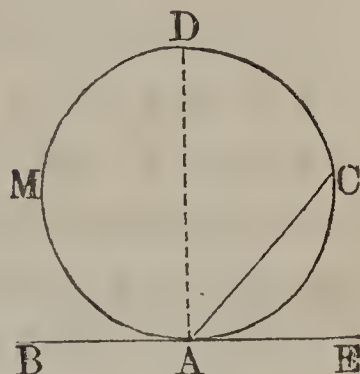


## PROPOSITION XXI. THEOREM.

*An angle formed by a tangent and a chord meeting it at the point of contact, is measured by half the included arc.*

Let  $BE$  be tangent to the circle  $AMC$ , and let  $AC$  be a chord drawn from the point of contact  $A$ : then will the angle  $BAC$  be measured by half of the arc  $AMC$ .

For, draw the diameter  $AD$ . The angle  $BAD$  is a right angle (P. IX.), and is measured by half the semi-circumference  $AMD$  (P. XVII., S.); the angle  $DAC$  is measured by half of the arc  $DC$  (P. XVIII.): hence, the angle  $BAC$ , which is equal to the sum of the angles  $BAD$  and  $DAC$ , is measured by half the sum of the arcs  $AMD$  and  $DC$ , or by half of the arc  $AMC$ ; which was to be proved.



The angle  $CAE$ , which is the difference of  $DAE$  and  $DAC$ , is measured by half the difference of the arcs  $DCA$  and  $DC$ , or by half the arc  $CA$ .



## PRACTICAL APPLICATIONS.

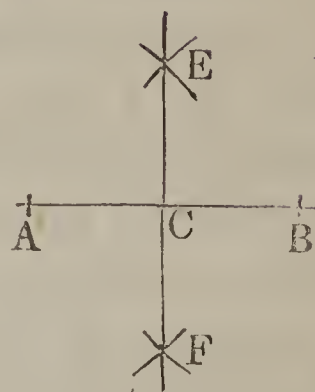
---

### PROBLEM I.

*To bisect a given straight line.*

Let  $AB$  be a given straight line.

From  $A$  and  $B$ , as centres, with a radius greater than one half of  $AB$ , describe arcs intersecting at  $E$  and  $F$ : join  $E$  and  $F$ , by the straight line  $EF$ . Then will  $EF$  bisect the given line  $AB$ . For,  $E$  and  $F$  are each equally distant from  $A$  and  $B$ ; and consequently, the line  $EF$  bisects  $AB$  (B. I., P. XVI., C.).

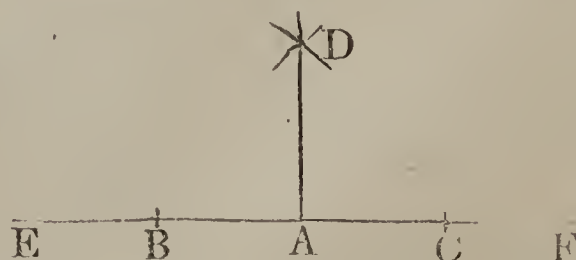


### PROBLEM II.

*To erect a perpendicular to a given straight line, at a given point of that line.*

Let  $EF$  be a given line, and let  $A$  be a given point on that line.

From  $A$ , lay off the equal distances  $AB$  and  $AC$ ; from  $B$  and  $C$ , as centres, with a radius greater than one half



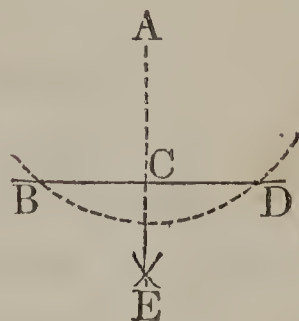
of  $BC$ , describe arcs intersecting at  $D$ ; draw the line  $AD$ : then will  $AD$  be the perpendicular required. For,  $D$  and  $A$  are each equally distant from  $B$  and  $C$ ; consequently,  $DA$  is perpendicular to  $BC$  (B. I., P. XVI., C.).

## PROBLEM III.

*To draw a perpendicular to a given straight line, from a given point without that line.*

Let  $BD$  be the given line, and  $A$  the given point.

From  $A$ , as a centre, with a radius sufficiently great, describe an arc cutting  $BD$  in two points,  $B$  and  $D$ ; with  $B$  and  $D$  as centres, and a radius greater than one-half of  $BD$ , describe arcs intersecting at  $E$ ; draw  $AE$ : then will  $AE$  be the perpendicular required. For,  $A$  and  $E$  are each equally distant from  $B$  and  $D$ : hence,  $AE$  is perpendicular to  $BD$  (B. I., P. XVI., C.).



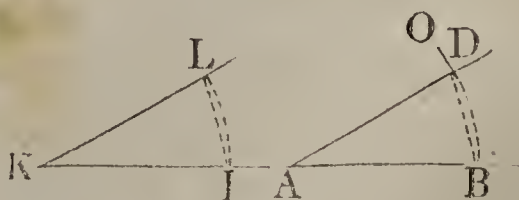
## PROBLEM IV.

*At a point on a given line, to construct an angle equal to a given angle.*

Let  $A$  be the given point,  $AB$  the given line, and  $IKL$  the given angle.

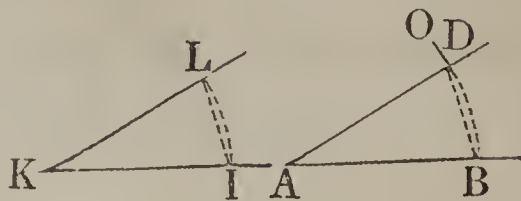
From the vertex  $K$  as a centre, with any radius  $KI$ , describe the arc  $IL$ , terminating in the sides of the angle.

From  $A$  as a centre, with a radius  $AB$ , equal to  $KI$ ,



describe the indefinite arc  $BO$ ; then, with a radius equal to the chord  $LI$ , from  $B$  as a centre, describe an arc cutting the arc  $BO$  in  $D$ ; draw  $AD$ : then will  $BAD$  be equal to the angle  $K$ .

For, the arcs  $BD$ ,  $IL$ , have equal radii and equal chords: hence, they are equal (P. IV.); therefore, the angles  $BAD$ ,  $IKL$ , measured by them, are also equal (P. XV.).

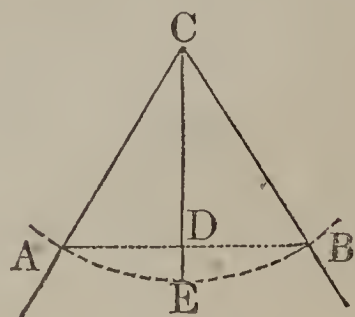


#### PROBLEM V.

*To bisect a given arc, or a given angle.*

1°. Let  $AEB$  be a given arc, and  $C$  its centre.

Draw the chord  $AB$ ; through  $C$ , draw  $CD$  perpendicular to  $AB$  (Prob. III.): then will  $CD$  bisect the arc  $AEB$  (P. VI.).



2°. Let  $ACB$  be a given angle.

With  $C$  as a centre, and any radius  $CB$ , describe the arc  $BA$ ; bisect it by the line  $CD$ , as just explained: then will  $CD$  bisect the angle  $ACB$ .

For, the arcs  $AE$  and  $EB$  are equal, from what was just shown; consequently, the angles  $ACE$  and  $ECB$  are also equal (P. XV.).

*Scholium.* If each half of an arc or angle be bisected, the original arc or angle will be divided into four equal parts; and if each of these be bisected, the original arc or angle will be divided into eight equal parts; and so on.

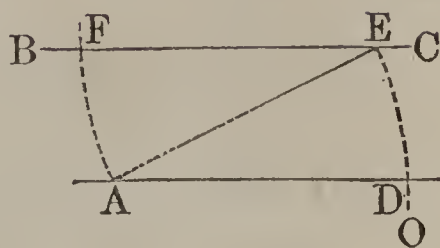


## PROBLEM VI.

*Through a given point, to draw a line parallel to a given line.*

Let  $A$  be a given point, and  $BC$  a given line.

From the point  $A$  as a centre, with a radius  $AE$ , greater than the shortest distance from  $A$  to  $BC$ , describe an indefinite arc  $EO$ ; from  $E$  as a centre, with the same radius, describe the arc  $AF$ ; lay off  $ED$  equal to  $AF$ , and draw  $AD$ : then will  $AD$  be the parallel required.



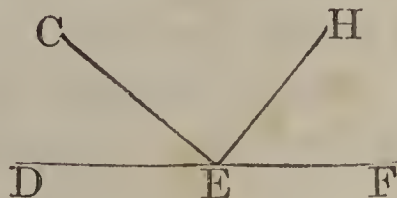
For, drawing  $AE$ , the angles  $AEF$ ,  $EAD$ , are equal (P. XV.); therefore, the lines  $AD$ ,  $EF$  are parallel (B. I., P. XIX., C. 1.).

## PROBLEM VII.

*Given, two angles of a triangle, to construct the third angle.*

Let  $A$  and  $B$  be given angles of a triangle.

Draw a line  $DF$ , and at some point of it, as  $E$ , construct the angle  $FEH$  equal to  $A$ , and  $HEC$  equal to  $B$ . Then, will  $CED$  be equal to the required angle.



For, the sum of the three angles at  $E$  is equal to two right angles (B. I., P. I., C. 3), as is also the sum of the three angles of a triangle (B. I., P. XXV.). Consequently, the third angle  $CED$  must be equal to the third angle of the triangle.

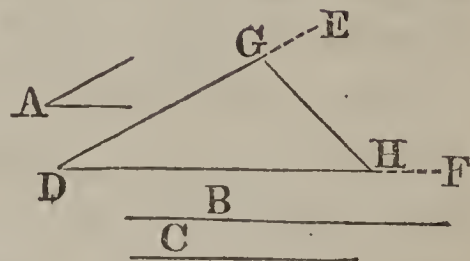
## PROBLEM VIII.

*Given, two sides and the included angle of a triangle, to construct the triangle.*

Let  $B$  and  $C$  denote the given sides, and  $A$  the given angle.

Draw the indefinite line  $DF$ , and at  $D$  construct an angle  $FDE$ , equal to the angle  $A$ ; on  $DF$ , lay off  $DH$  equal to the side  $C$ , and on  $DE$ , lay off  $DG$  equal to the side  $B$ ; draw

$GH$ : then will  $DGH$  be the required triangle (B. I., P. V.).



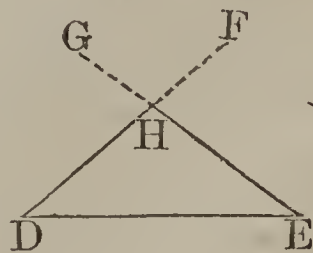
## PROBLEM IX.

*Given, one side and two angles of a triangle, to construct the triangle.*

The two angles may be either both adjacent to the given side, or one may be adjacent and the other opposite to it. In the latter case, construct the third angle by Problem VII. We shall then have two angles and their included side.

Draw a straight line, and on it lay off  $DE$  equal to the given side; at  $D$  construct an angle equal to one of the adjacent angles, and at  $E$  construct an angle equal to the other adjacent angle;

produce the sides  $DF$  and  $EG$  till they intersect at  $H$ : then will  $DEH$  be the triangle required (B. I., P. VI.).

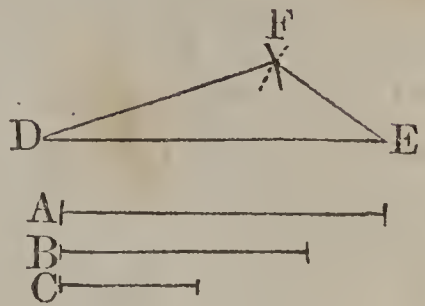


PROBLEM X.

*Given, the three sides of a triangle, to construct the triangle.*

Let  $A$ ,  $B$ , and  $C$ , be the given sides.

Draw  $DE$ , and make it equal to the side  $A$ ; from  $D$  as a centre, with a radius equal to the side  $B$ , describe an arc; from  $E$  as a centre, with a radius equal to the side  $C$ , describe an arc intersecting the former at  $F$ ; draw  $DF$  and  $EF$ : then will  $DEF$  be the triangle required (B. I., P. X.).



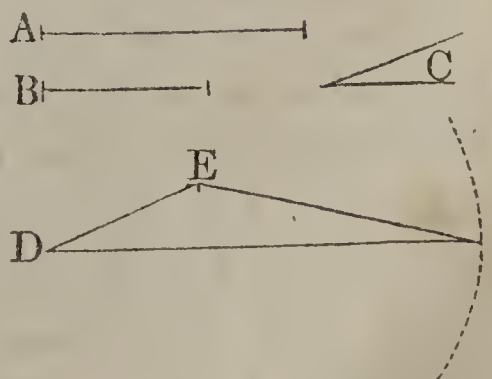
*Scholium.* In order that the construction may be possible, any one of the given sides must be less than the sum of the other two, and greater than their difference (B. I., P. VII., S.).

PROBLEM XI.

*Given, two sides of a triangle, and the angle opposite one of them, to construct the triangle.*

Let  $A$  and  $B$  be the given sides, and  $C$  the given angle.

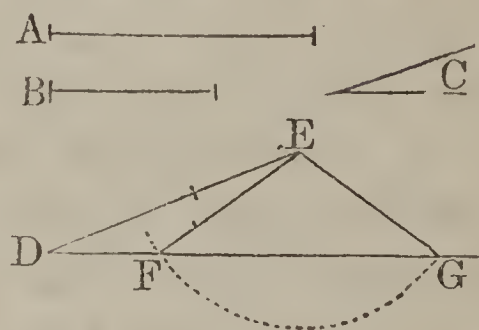
Draw an indefinite line  $DG$ , and at some point of it, as  $D$ , construct an angle  $GDE$  equal to the given angle; on one side of this angle lay off the distance  $DE$  equal to the side  $B$  adjacent to the given angle; from  $E$  as a centre, with a radius equal to the side opposite the given angle, describe an arc cutting the side  $DG$  at  $G$ ; draw  $EG$ . Then will  $DEG$  be the required triangle.





For, the sides  $DE$  and  $EG$  are equal to the given sides, and the angle  $D$ , opposite one of them, is equal to the given angle.

*Scholium.* When the side opposite the given angle is greater than the other given side, there will be but one solution. When the given angle is acute, and the side opposite the given angle is less than the other given side, and greater than the shortest distance from  $E$  to  $DG$ , there will be two solutions,  $DEG$  and  $DEF$ . When the side opposite the given angle is equal to the shortest distance from  $E$  to  $DG$ , the arc will be tangent to  $DG$ , the angle opposite  $DE$  will be a right angle, and there will be but one solution. When the side opposite the given angle is shorter than the distance from  $E$  to  $DG$ , there will be no solution.

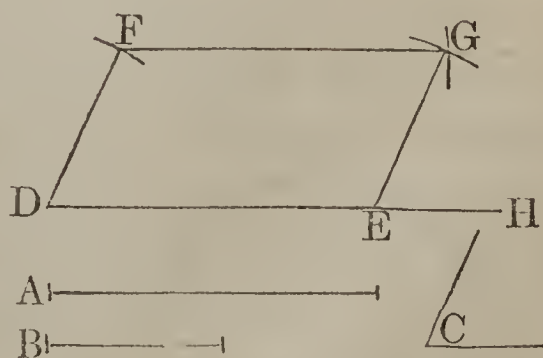


### PROBLEM XII.

*Given, two adjacent sides of a parallelogram and their included angle, to construct the parallelogram.*

Let  $A$  and  $B$  be the given sides, and  $C$  the given angle.

Draw the line  $DH$ , and at some point as  $D$ , construct the angle  $HDF$  equal to the angle  $C$ . Lay off  $DE$  equal to the side  $A$ , and  $DF$  equal to the side  $B$ ; draw  $FG$  parallel to  $DE$ , and  $EG$  parallel to  $DF$ : then will  $DFGE$  be the parallelogram required.

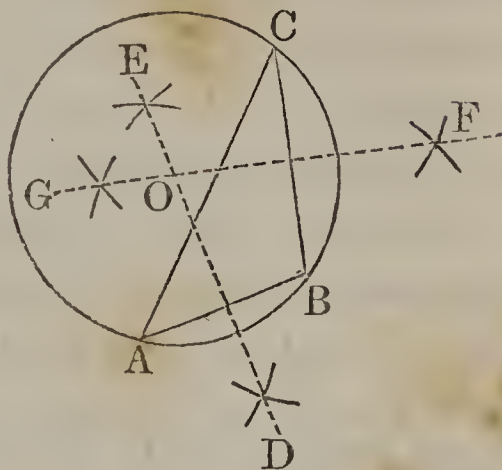


For, the opposite sides are parallel by construction; and consequently, the figure is a parallelogram (D. 28); it is also formed with the given sides and given angle.

## PROBLEM XIII.

*To find the centre of a given circumference.*

Take any three points  $A$ ,  $B$ , and  $C$ , on the circumference or arc, and join them by the chords  $AB$ ,  $BC$ ; bisect these chords by the perpendiculars  $DE$  and  $FG$ : then will their point of intersection  $O$ , be the centre required (P. VII.).



*Scholium.* The same construction enables us to pass a circumference through any three points not in a straight line. If the points are vertices of a triangle, the circle will be circumscribed about it.

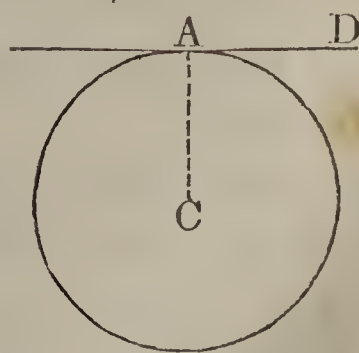
## PROBLEM XIV.

*Through a given point, to draw a tangent to a given circle.*

There may be two cases: the given point may lie on the circumference of the given circle, or it may lie without the given circle.

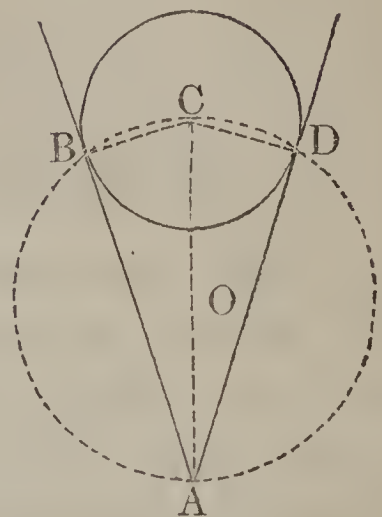
1°. Let  $C$  be the centre of the given circle, and  $A$  a point on the circumference, through which the tangent is to be drawn.

Draw the radius  $CA$ , and at  $A$  draw  $AD$  perpendicular to  $AC$ : then will  $AD$  be the tangent required (P. IX.).



2°. Let  $C$  be the centre of the given circle, and  $A$  a point without the circle, through which the tangent is to be drawn.

Draw the line  $AC$ ; bisect it at  $O$ , and from  $O$  as a centre, with a radius  $OC$ , describe the circumference  $ABCD$ ; join the point  $A$  with the points of intersection  $D$  and  $B$ : then will both  $AD$  and  $AB$  be tangent to the given circle, and there will be two solutions.



For, the angles  $ABC$  and  $ADC$  are right angles (P. XVIII., C. 2): hence, each of the lines  $AB$  and  $AD$  is perpendicular to a radius at its extremity; and consequently, they are tangent to the given circle (P. IX.).

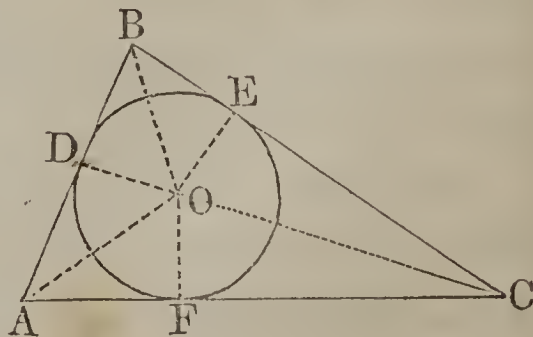
*Corollary.* The right-angled triangles  $ABC$  and  $ADC$ , have a common hypotenuse  $AC$ , and the side  $BC$  equal to  $DC$ ; and consequently, they are equal in all their parts (B. I., P. XVII.): hence,  $AB$  is equal to  $AD$ , and the angle  $CAB$  is equal to the angle  $CAD$ . The tangents are therefore equal, and the line  $AC$  bisects the angle between them.

#### PROBLEM XV.

*To inscribe a circle in a given triangle.*

Let  $ABC$  be the given triangle.

Bisect the angles  $A$  and  $B$ , by the lines  $AO$  and  $BO$ , meeting in the point  $O$  (Prob. V.); from the point  $O$





let fall the perpendiculars  $OD$ ,  $OE$ ,  $OF$ , on the sides of the triangle: these perpendiculars will all be equal.

For, in the triangles  $BOD$  and  $BOE$ , the angles  $OBE$  and  $OBD$  are equal, by construction; the angles  $ODB$  and  $OEB$  are equal, because both are right angles; and consequently, the angles  $BOD$  and  $BOE$  are also equal (B. I., P. XXV., C. 2), and the side  $OB$  is common; and therefore, the triangles are equal in all their parts (B. I., P. VI.): hence,  $OD$  is equal to  $OE$ . In like manner, it may be shown that  $OD$  is equal to  $OF$ .

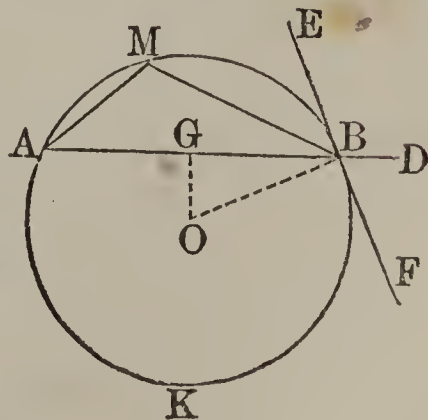
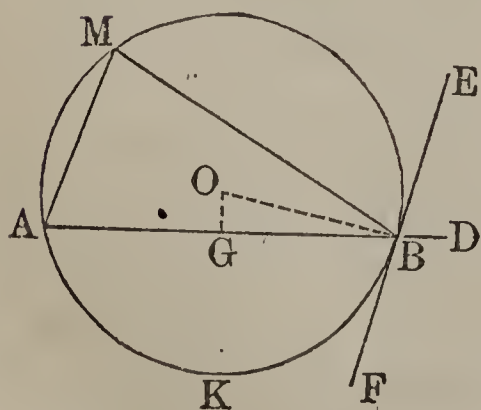
From  $O$  as a centre, with a radius  $OD$ , describe a circle, and it will be the circle required. For, each side is perpendicular to a radius at its extremity, and is therefore tangent to the circle.

*Corollary.* The lines that bisect the three angles of a triangle all meet in one point.

## PROBLEM XVI.

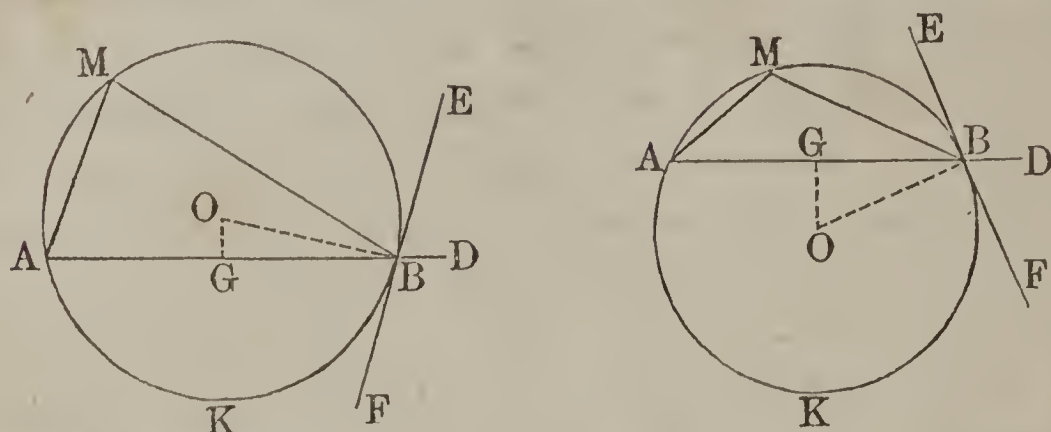
*On a given line, to construct a segment that shall contain a given angle.*

Let  $AB$  be the given line.



Produce  $AB$  towards  $D$ ; at  $B$  construct the angle  $DBE$  equal to the given angle; draw  $BO$  perpendicular

to  $BE$ , and at the middle point  $G$ , of  $AB$ , draw  $GO$  perpendicular to  $AB$ ; from their point of intersection  $O$ , as a centre, with a radius  $OB$ , describe the arc  $AMB$ : then will the segment  $AMB$  be the segment required.



For, the angle  $ABF$ , equal to  $EBD$ , is measured by half of the arc  $AKB$  (P. XXI.); and the inscribed angle  $AMB$  is measured by half of the same arc: hence, the angle  $AMB$  is equal to the angle  $EBD$ , and consequently, to the given angle.

## BOOK IV.

### MEASUREMENT AND RELATION OF POLYGONS.

#### DEFINITIONS.

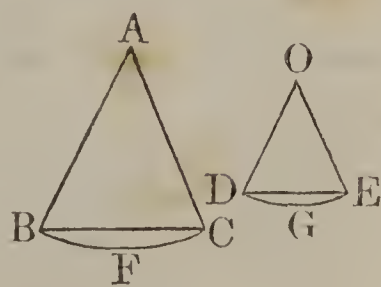
1. **SIMILAR POLYGONS**, are polygons which are mutually equiangular, and which have the sides about the equal angles, taken in the same order, proportional.

2. In similar polygons, the parts which are similarly placed in each, are called *homologous*.

The corresponding angles are *homologous angles*, the corresponding sides are *homologous sides*, the corresponding diagonals are *homologous diagonals*, and so on.

3. **SIMILAR ARCS, SECTORS, or SEGMENTS** are those which correspond to equal angles at the centre.

Thus, if the angles  $A$  and  $O$  are equal, the arcs  $BFC$  and  $DGE$  are similar, the sectors  $BAC$  and  $DOE$  are similar, and the segments  $BFC$  and  $DGE$  are similar.



4. The **ALTITUDE OF A TRIANGLE**, is the perpendicular distance from the vertex of either angle to the opposite side, or the opposite side produced.

The vertex of the angle from which the distance is measured, is called the *vertex of the triangle*, and the opposite side, is called the *base of the triangle*.





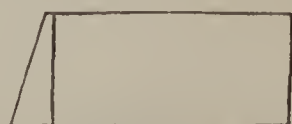
5. The ALTITUDE OF A PARALLELOGRAM, is the perpendicular distance between two opposite sides.

These sides are called *bases* ; one the *upper*, and the other, the *lower base*.



6. The ALTITUDE OF A TRAPEZOID, is the perpendicular distance between its parallel sides.

These sides are called *bases* ; one the *upper*, and the other, the *lower base*.



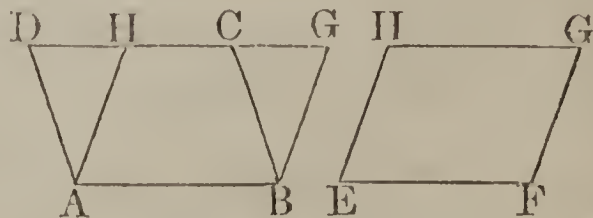
7. The AREA OF A SURFACE, is its numerical value expressed in terms of some other surface taken as a *unit*. The unit adopted is a square described on the linear unit, as a side.

#### PROPOSITION I. THEOREM.

*Parallelograms which have equal bases and equal altitudes, are equal.*

Let the parallelograms  $ABCD$  and  $EFGH$  have equal bases and equal altitudes : then will the parallelograms be equal.

For, let them be so placed that their lower bases shall coincide ; then, because they have the same altitude, their upper bases will be in the same line  $DG$ , parallel to  $AB$ .



The triangles  $DAH$  and  $CBG$ , have the sides  $AD$  and  $BC$  equal, because they are opposite sides of the parallelogram  $AC$  (B. I., P. XXVIII.) ; the sides  $AH$  and  $BG$  equal, because they are opposite sides of the parallelogram  $AG$  ; the angles  $DAH$  and  $CBG$  equal, because their

sides are parallel and lie in the same direction (B. I., P. XXIV.) : hence, the triangles are equal (B. I., P. V.).

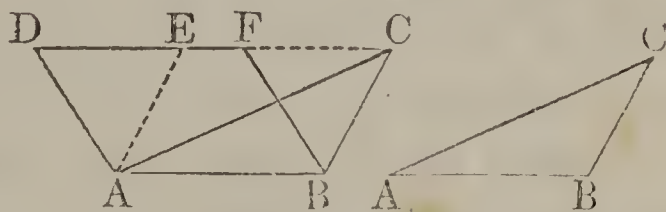
If from the quadrilateral  $ABGD$ , we take away the triangle  $DAH$ , there will remain the parallelogram  $AG$ ; if from the same quadrilateral  $ABGD$ , we take away the triangle  $CBG$ , there will remain the parallelogram  $AC$ : hence, the parallelogram  $AC$  is equal to the parallelogram  $EG$  (A. 3); *which was to be proved.*

### PROPOSITION II. THEOREM.

*A triangle is equal to one-half of a parallelogram having an equal base and an equal altitude.*

Let the triangle  $ABC$ , and the parallelogram  $ABFD$ , have equal bases and equal altitudes: then will the triangle be equal to one-half of the parallelogram.

For, let them be so placed that the base of the triangle shall coincide with the lower base of the parallelogram;



then, because they have equal altitudes, the vertex of the triangle will lie in the upper base of the parallelogram, or in the prolongation of that base.

From  $A$ , draw  $AE$  parallel to  $BC$ , forming the parallelogram  $ABCE$ . This parallelogram will be equal to the parallelogram  $ABFD$ , from Proposition I. But the triangle  $ABC$  is equal to half of the parallelogram  $ABCE$  (B. I., P. XXVIII., C. 1): hence, it is equal to half of the parallelogram  $ABFD$  (A. 7); *which was to be proved.*

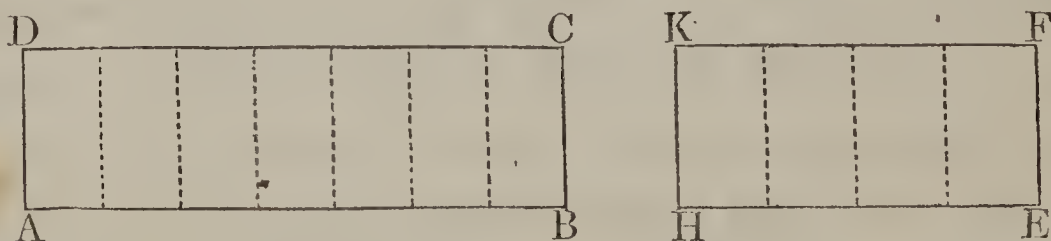
*Cor.* Triangles having equal bases and equal altitudes are equal, for they are halves of equal parallelograms.

## PROPOSITION III. THEOREM.

*Rectangles having equal altitudes, are proportional to their bases.*

There may be two cases: the bases may be commensurable, or they may be incommensurable.

1°. Let  $ABCD$  and  $HEFK$ , be two rectangles whose altitudes  $AD$  and  $HK$  are equal, and whose bases  $AB$  and  $HE$  are commensurable: then will the areas of the rectangles be proportional to their bases.



Suppose that  $AB$  is to  $HE$ , as 7 is to 4. Conceive  $AB$  to be divided into 7 equal parts, and  $HE$  into 4 equal parts, and at the points of division, let perpendiculars be drawn to  $AB$  and  $HE$ . Then will  $ABCD$  be divided into 7, and  $HEFK$  into 4 rectangles, all of which will be equal, because they have equal bases and equal altitudes (P. I.): hence, we have,

$$ABCD : HEFK :: 7 : 4.$$

But we have, by hypothesis,

$$AB : HE :: 7 : 4.$$

From these proportions, we have (B. II., P. IV.),

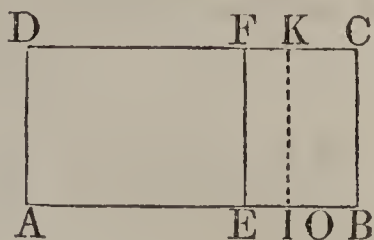
$$ABCD : HEFK :: AB : HE.$$

Had any other numbers than 7 and 4 been used, the same proportion would have been found; *which was to be proved.*



2°. Let the bases of the rectangles be incommensurable : then will the rectangles be proportional to their bases.

For, place the rectangle *HEFK* upon the rectangle *ABCD*, so that it shall take the position *AEFD*. Then, if the rectangles are not proportional to their bases, let us suppose that



$$ABCD : AEFD :: AB : AO;$$

in which *AO* is greater than *AE*. Divide *AB* into equal parts, each less than *OE*; at least one point of division, as *I*, will fall between *E* and *O*; at this point, draw *IK* perpendicular to *AB*. Then, because *AB* and *AI* are commensurable, we shall have, from what has just been shown,

$$ABCD : AIKD :: AB : AI.$$

The above proportions have their antecedents the same in each; hence (B. II., P. IV., C.),

$$AEFD : AIKD :: AO : AI.$$

The rectangle *AEFD* is less than *AIKD*; and if the above proportion were true, the line *AO* would be less than *AI*; whereas, it is greater. The fourth term of the proportion, therefore, cannot be greater than *AE*. In like manner, it may be shown that it cannot be less than *AE*; consequently, it must be equal to *AE*: hence,

$$ABCD : AEFD :: AB : AE;$$

*which was to be proved.*

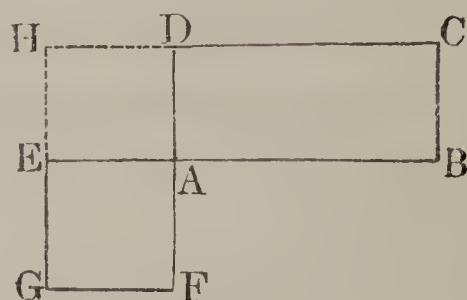
*Cor.* If rectangles have equal bases, they are to each other as their altitudes.

## PROPOSITION IV. THEOREM.

*Any two rectangles are to each other as the products of their bases and altitudes.*

Let  $ABCD$  and  $AEGF$  be two rectangles: then will  $ABCD$  be to  $AEGF$ , as  $AB \times AD$  is to  $AE \times AF$ .

For, place the rectangles so that the angles  $DAB$  and  $EAF$  shall be opposite or vertical; then, produce the sides  $CD$  and  $GE$  till they meet in  $H$ .



The rectangles  $ABCD$  and  $ADHE$  have the same altitude  $AD$ : hence (P. III.),

$$ABCD : ADHE :: AB : AE.$$

The rectangles  $ADHE$  and  $AEGF$  have the same altitude  $AE$ : hence,

$$ADHE : AEGF :: AD : AF.$$

Multiplying these proportions, term by term (B. II., P. XII.), and omitting the common factor  $ADHE$  (B. II., P. VII.), we have,

$$ABCD : AEGF :: AB \times AD : AE \times AF;$$

*which was to be proved.*

*Scholium 1.* If we suppose  $AE$  and  $AF$ , each to be equal to the linear unit, the rectangle  $AEGF$  will be the superficial unit, and we shall have,

$$ABCD : 1 :: AB \times AD : 1;$$

$$ABCD = AB \times AD :$$

hence, *the area of a rectangle is equal to the product of its base and altitude*; that is, the number of superficial units in the rectangle, is equal to the product of the number of linear units in its base by the number of linear units in its altitude.

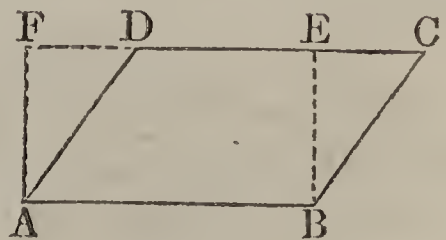
*Scholium 2.* The product of two lines is sometimes called the *rectangle* of the lines, because the product is equal to the area of a rectangle constructed with the lines as sides.

PROPOSITION V. THEOREM.

*The area of a parallelogram is equal to the product of its base and altitude.*

Let  $ABCD$  be a parallelogram,  $AB$  its base, and  $BE$  its altitude: then will the area of  $ABCD$  be equal to  $AB \times BE$ .

For, construct the rectangle  $ABEF$ , having the same base and altitude: then will the rectangle be equal to the parallelogram (P. I.); but the area of the rectangle is equal to  $AB \times BE$ :



hence, the area of the parallelogram is also equal to  $AB \times BE$ ; which was to be proved.

*Cor.* Parallelograms are to each other as the products of their bases and altitudes. If their altitudes are equal, they are to each other as their bases. If their bases are equal, they are to each other as their altitudes.

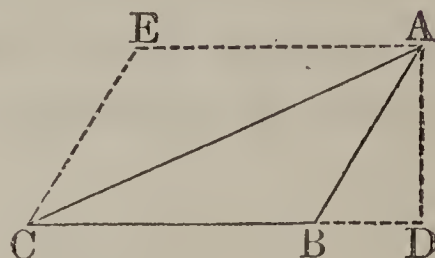


## PROPOSITION VI. THEOREM.

*The area of a triangle is equal to half the product of its base and altitude.*

Let  $ABC$  be a triangle,  $BC$  its base, and  $AD$  its altitude: then will the area of the triangle be equal to  $\frac{1}{2}BC \times AD$ .

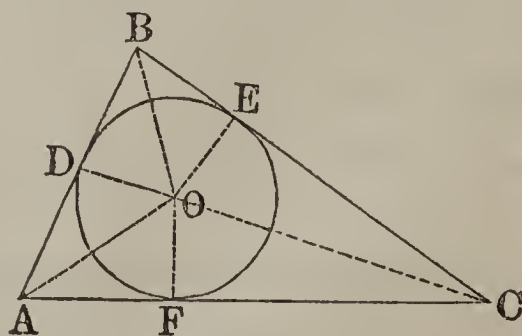
For, from  $C$ , draw  $CE$  parallel to  $BA$ , and from  $A$ , draw  $AE$  parallel to  $CB$ . The area of the parallelogram  $BCEA$  is  $BC \times AD$  (P. V.); but the triangle  $ABC$  is half of the parallelogram  $BCEA$ : hence, its area is equal to  $\frac{1}{2}BC \times AD$ ; which was to be proved.



*Cor. 1.* Triangles are to each other, as the products of their bases and altitudes (B. II., P. VII.). If their altitudes are equal, they are to each other as their bases. If their bases are equal, they are to each other as their altitudes.

*Cor. 2.* The area of a triangle is equal to half the product of its perimeter and the radius of the inscribed circle.

For, let  $DEF$  be a circle inscribed in the triangle  $ABC$ . Draw  $OD$ ,  $OE$ , and  $OF$ , to the points of contact, and  $OA$ ,  $OB$ , and  $OC$ , to the vertices.



The area of  $OBC$  will be equal to  $\frac{1}{2}OE \times BC$ ; the area of  $OAC$  will be equal to  $\frac{1}{2}OF \times AC$ ; and the area

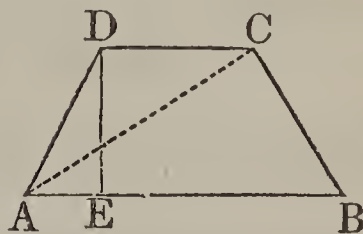
of  $OAB$  will be equal to  $\frac{1}{2}OD \times AB$ ; and since  $OD$ ,  $OE$ , and  $OF$ , are equal, the area of the triangle  $ABC$  (A. 9), will be equal to  $\frac{1}{2}OD (AB + BC + CA)$ .

PROPOSITION VII. THEOREM.

*The area of a trapezoid is equal to the product of its altitude and half the sum of its parallel sides.*

Let  $ABCD$  be a trapezoid,  $DE$  its altitude, and  $AB$  and  $DC$  its parallel sides: then will its area be equal to  $DE \times \frac{1}{2}(AB + DC)$ .

For, draw the diagonal  $AC$ , forming the triangles  $ABC$  and  $ACD$ . The altitude of each of these triangles is equal to  $DE$ . The area of  $ABC$  is equal to  $\frac{1}{2}AB \times DE$  (P. VI.); the area of  $ACD$  is equal to  $\frac{1}{2}DC \times DE$ : hence, the area of the trapezoid, which is the sum of the triangles, is equal to the sum of  $\frac{1}{2}AB \times DE$  and  $\frac{1}{2}DC \times DE$ , or to  $DE \times \frac{1}{2}(AB + DC)$ ; which was to be proved.



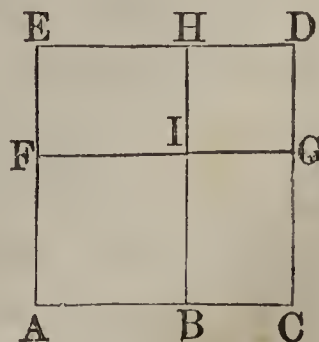
PROPOSITION VIII. THEOREM.

*The square described on the sum of two lines is equal to the sum of the squares described on the lines, increased by twice the rectangle of the lines.*

Let  $AB$  and  $BC$  be two lines, and  $AC$  their sum: then will

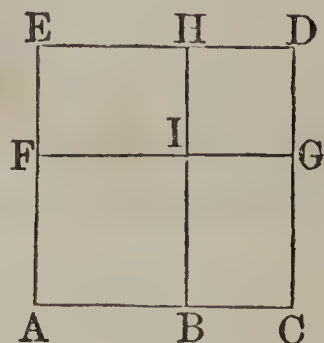
$$\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2 + 2AB \times BC.$$

On  $AC$ , construct the square  $ACDE$ ; from  $B$ , draw  $BH$  par-



allel to  $AE$ ; lay off  $AF$  equal to  $AB$ , and from  $F$ , draw  $FG$  parallel to  $AC$ : then will  $IG$  and  $IH$  be each equal to  $BC$ ; and  $IB$  and  $IF$ , to  $AB$ .

The square  $ACDE$  is composed of four parts. The part  $ABIF$  is a square described on  $AB$ ; the part  $IGDH$  is equal to a square described on  $BC$ ; the part  $BCGI$  is equal to the rectangle of  $AB$  and  $BC$ ; and the part  $FIHE$  is also equal to the rectangle of  $AB$  and  $BC$ : and because the whole is equal to the sum of all its parts (A. 9), we have,



$$\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2 + 2AB \times BC;$$

*which was to be proved.*

*Cor.* If the lines  $AB$  and  $BC$  are equal, the four parts of the square on  $AC$  will also be equal: hence, *the square described on a line is equal to four times the square described on half the line.*

#### PROPOSITION IX. THEOREM.

*The square described on the difference of two lines is equal to the sum of the squares described on the lines, diminished by twice the rectangle of the lines.*

Let  $AB$  and  $BC$  be two lines, and  $AC$  their difference: then will

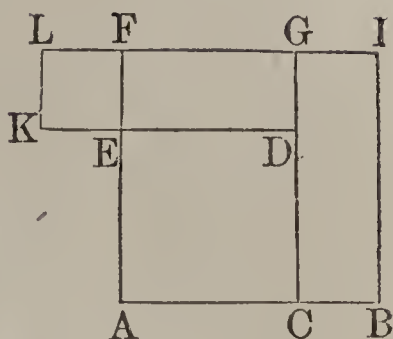
$$\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2 - 2AB \times BC.$$

On  $AB$  construct the square  $ABIF$ ; from  $C$  draw  $CG$  parallel to  $BI$ ; lay off  $CD$  equal to  $AC$ , and from  $D$  draw  $DK$  parallel and equal to  $BA$ ; complete



the square  $EFLK$ : then will  $EK$  be equal to  $BC$ , and  $EFLK$  will be equal to the square of  $BC$ .

The whole figure  $ABILKE$  is equal to the sum of the squares described on  $AB$  and  $BC$ . The part  $CBIG$  is equal to the rectangle of  $AB$  and  $BC$ ; the part  $DGLK$  is also equal to the rectangle of  $AB$  and  $BC$ . If from the whole figure  $ABILKE$ , the two parts  $CBIG$  and  $DGLK$  be taken, there will remain the part  $ACDE$ , which is equal to the square of  $AC$ : hence,



$$\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2 - 2AB \times BC;$$

which was to be proved.

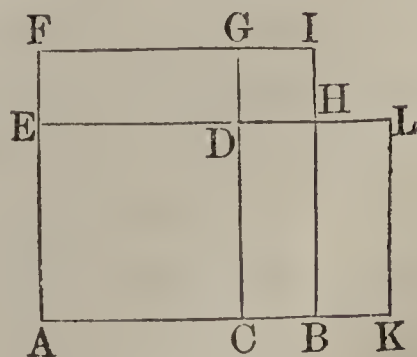
#### PROPOSITION X. THEOREM.

*The rectangle contained by the sum and difference of two lines, is equal to the difference of their squares.*

Let  $AB$  and  $BC$  be two lines, of which  $AB$  is the greater: then will

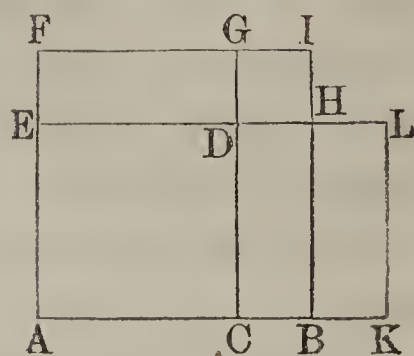
$$(AB + BC)(AB - BC) = \overline{AB}^2 - \overline{BC}^2.$$

On  $AB$ , construct the square  $ABIF$ ; prolong  $AB$ , and make  $BK$  equal to  $BC$ ; then will  $AK$  be equal to  $AB + BC$ ; from  $K$ , draw  $KL$  parallel to  $BI$ , and make it equal to  $AC$ ; draw  $LE$  parallel to  $KA$ , and  $CG$  parallel to  $BI$ : then  $DG$  is equal to



$BC$ , and the figure  $DHIG$  is equal to the square on  $BC$ , and  $EDGF$  is equal to  $BKLH$ .

If we add to the figure  $ABHE$ , the rectangle  $BKLH$ , we shall have the rectangle  $AKLE$ , which is equal to the rectangle of  $AB + BC$  and  $AB - BC$ . If to the same figure  $ABHE$ , we add the rectangle  $DGFE$ , equal to  $BKLH$ , we shall have the figure  $ABHDGF$ , which is equal to the difference of the squares of  $AB$  and  $BC$ . But the sums of equals are equal (A. 2), hence,



$$(AB + BC)(AB - BC) = \overline{AB}^2 - \overline{BC}^2;$$

*which was to be proved.*

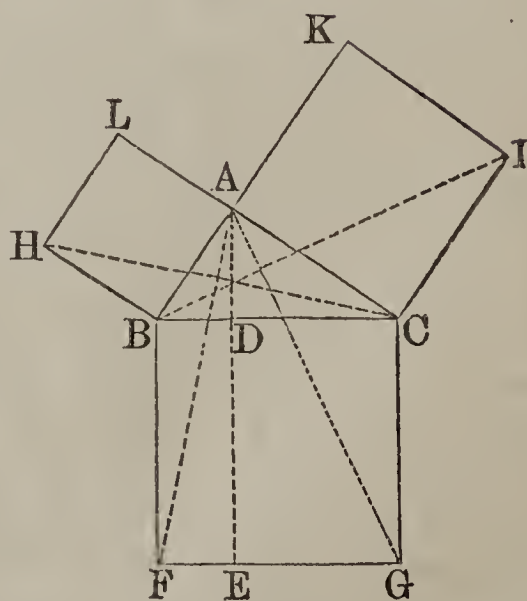
#### PROPOSITION XI. THEOREM.

*The square described on the hypotenuse of a right-angled triangle, is equal to the sum of the squares described on the other two sides.*

Let  $ABC$  be a triangle, right-angled at  $A$ : then will  $\overline{BC}^2 = \overline{AB}^2 + \overline{AC}^2$ .

Construct the square  $BG$  on the side  $BC$ , the square  $AH$  on the side  $AB$ , and the square  $AI$  on the side  $AC$ ; from  $A$  draw  $AD$  perpendicular to  $BC$ , and prolong it to  $E$ : then will  $DE$  be parallel to  $BF$ ; draw  $AF$  and  $HC$ .

In the triangles  $HBC$  and  $ABF$ , we have  $HB$  equal to  $AB$ , because they are sides of the same square;



$BC$  equal to  $BF$ , for the same reason, and the included angles  $HBC$  and  $ABF$  equal, because each is equal to the angle  $ABC$  plus a right angle: hence, the triangles are equal in all their parts (B. I., P. V.).

The triangle  $ABF$ , and the rectangle  $BE$ , have the same base  $BF$ , and because  $DE$  is the prolongation of  $DA$ , their altitudes are equal: hence, the triangle  $ABF$  is equal to half the rectangle  $BE$  (P. II.). The triangle  $HBC$ , and the square  $BL$ , have the same base  $BH$ , and because  $AC$  is the prolongation of  $AL$  (B. I., P. IV.), their altitudes are equal: hence, the triangle  $HBC$  is equal to half the square of  $AH$ . But, the triangles  $ABF$  and  $HBC$  are equal: hence, the rectangle  $BE$  is equal to the square  $AH$ . In the same manner, it may be shown that the rectangle  $DG$  is equal to the square  $AI$ : hence, the sum of the rectangles  $BE$  and  $DG$ , or the square  $BG$ , is equal to the sum of the squares  $AH$  and  $AI$ ; or,  $\overline{BC}^2 = \overline{AB}^2 + \overline{AC}^2$ ; which was to be proved.

*Cor. 1.* The square of either side about the right angle is equal to the square of the hypotenuse diminished by the square of the other side: thus,

$$\overline{AB}^2 = \overline{BC}^2 - \overline{AC}^2; \quad \text{or,} \quad \overline{AC}^2 = \overline{BC}^2 - \overline{AB}^2.$$

*Cor. 2.* If from the vertex of the right angle, a perpendicular be drawn to the hypotenuse, dividing it into two segments,  $BD$  and  $DC$ , the square of the hypotenuse will be to the square of either of the other sides, as the hypotenuse is to the segment adjacent to that side.

For, the square  $BG$ , is to the rectangle  $BE$ , as  $BC$  to  $BD$  (P. III.); but the rectangle  $BE$  is equal to the square  $AH$ : hence,

$$\overline{BC}^2 : \overline{AB}^2 :: BC : BD.$$

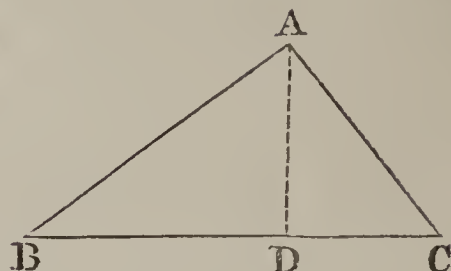


In like manner, we have,

$$\overline{BC}^2 : \overline{AC}^2 :: BC : DC.$$

*Cor. 3. The squares of the sides about the right angle are to each other as the adjacent segments of the hypotenuse.*

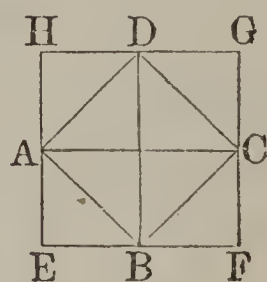
For, by combining the proportions of the preceding corollary (B. II., P. IV., C.), we have,



$$\overline{AB}^2 : \overline{AC}^2 :: BD : DC.$$

*Cor. 4. The square described on the diagonal of a square is double the given square.*

For, the square of the diagonal is equal to the sum of the squares of the two sides; but the square of each side is equal to the given square: hence,



$$\overline{AC}^2 = 2\overline{AB}^2; \quad \text{or,} \quad \overline{AC}^2 = 2\overline{BC}^2.$$

*Cor. 5. From the last corollary, we have,*

$$\overline{AC}^2 : \overline{AB}^2 :: 2 : 1;$$

hence, by extracting the square root of each term, we have,

$$AC : AB :: \sqrt{2} : 1;$$

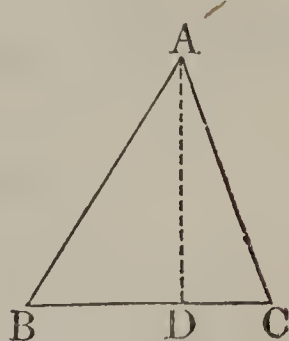
that is, the diagonal of a square is to the side, as the square root of two to one; consequently, the diagonal and the side of a square are incommensurable.

## PROPOSITION XII. THEOREM.

*In any triangle, the square of a side opposite an acute angle, is equal to the sum of the squares of the base and the other side, diminished by twice the rectangle of the base and the distance from the vertex of the acute angle to the foot of the perpendicular drawn from the vertex of the opposite angle to the base, or to the base produced.*

Let  $ABC$  be a triangle,  $C$  one of its acute angles,  $BC$  its base, and  $AD$  the perpendicular drawn from  $A$  to  $BC$ , or  $BC$  produced; then will

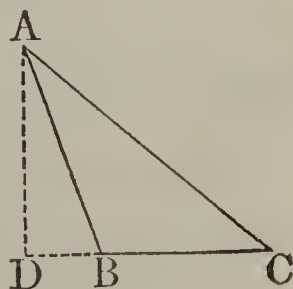
$$\overline{AB}^2 = \overline{BC}^2 + \overline{AC}^2 - 2BC \times CD.$$



For, whether the perpendicular meets the base, or the base produced, we have  $BD$  equal to the difference of  $BC$  and  $CD$ : hence (P. IX.),

$$\overline{BD}^2 = \overline{BC}^2 + \overline{CD}^2 - 2BC \times CD.$$

Adding  $\overline{AD}^2$  to both members, we have,



$$\overline{BD}^2 + \overline{AD}^2 = \overline{BC}^2 + \overline{CD}^2 + \overline{AD}^2 - 2BC \times CD.$$

But,  $\overline{BD}^2 + \overline{AD}^2 = \overline{AB}^2$ , and  $\overline{CD}^2 + \overline{AD}^2 = \overline{AC}^2$ :

hence,

$$\overline{AB}^2 = \overline{BC}^2 + \overline{AC}^2 - 2BC \times CD;$$

*which was to be proved.*

## PROPOSITION XIII. THEOREM.

*In any obtuse-angled triangle, the square of the side opposite the obtuse angle is equal to the sum of the squares of the base and the other side, increased by twice the rectangle of the base and the distance from the vertex of the obtuse angle to the foot of the perpendicular drawn from the vertex of the opposite angle to the base produced.*

Let  $ABC$  be an obtuse-angled triangle,  $B$  its obtuse angle,  $BC$  its base, and  $AD$  the perpendicular drawn from  $A$  to  $BC$  produced; then will

$$\overline{AC}^2 = \overline{BC}^2 + \overline{AB}^2 + 2BC \times BD.$$

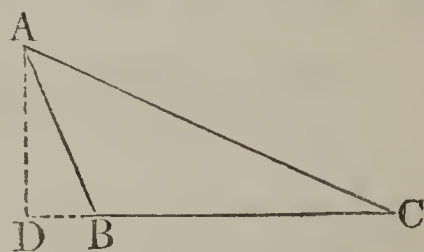
For,  $CD$  is the sum of  $BC$  and  $BD$ : hence (P. VIII.),

$$\overline{CD}^2 = \overline{BC}^2 + \overline{BD}^2 + 2BC \times BD.$$

Adding  $\overline{AD}^2$  to both members, and reducing, we have,

$$\overline{AC}^2 = \overline{BC}^2 + \overline{AB}^2 + 2BC \times BD;$$

*which was to be proved.*



*Scholium.* The right-angled triangle is the only one in which the sum of the squares described on two sides is equal to the square described on the third side.

## PROPOSITION XIV. THEOREM.

*In any triangle, the sum of the squares described on two sides is equal to twice the square of half the third side, increased by twice the square of the line drawn from the middle point of that side to the vertex of the opposite angle.*

Let  $ABC$  be any triangle, and  $EA$  a line drawn from



the middle of the base  $BC$  to the vertex  $A$  : then will

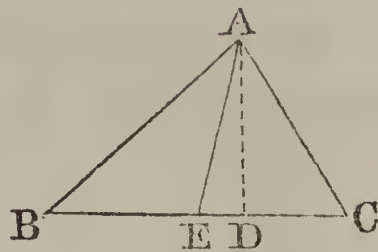
$$\overline{AB}^2 + \overline{AC}^2 = 2\overline{BE}^2 + 2\overline{EA}^2.$$

Draw  $AD$  perpendicular to  $BC$  ; then, from Proposition XII. we have,

$$\overline{AC}^2 = \overline{EC}^2 + \overline{EA}^2 - 2EC \times ED.$$

From Proposition XIII., we have,

$$\overline{AB}^2 = \overline{BE}^2 + \overline{EA}^2 + 2BE \times ED.$$

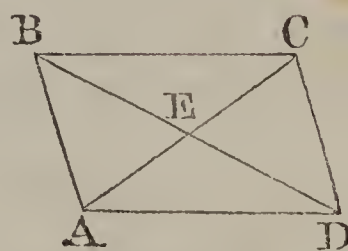


Adding these equations, member to member (A. 2), recollecting that  $BE$  is equal to  $EC$ , we have,

$$\overline{AB}^2 + \overline{AC}^2 = 2\overline{BE}^2 + 2\overline{EA}^2 ;$$

which was to be proved.

*Cor.* Let  $ABCD$  be a parallelogram, and  $BD$ ,  $AC$ , its diagonals. Then, since the diagonals mutually bisect each other (B. I., P. XXXI.), we shall have,



$$\begin{aligned} \overline{AB}^2 + \overline{BC}^2 &= 2\overline{AE}^2 + 2\overline{BE}^2 ; \\ \text{and,} \quad \overline{CD}^2 + \overline{DA}^2 &= 2\overline{CE}^2 + 2\overline{DE}^2 , \end{aligned}$$

whence, by addition, recollecting that  $AE$  is equal to  $CE$ , and  $BE$  to  $DE$ , we have,

$$\overline{AB}^2 + \overline{BC}^2 + \overline{CD}^2 + \overline{DA}^2 = 4\overline{CE}^2 + 4\overline{DE}^2 ;$$

but,  $4\overline{CE}^2$  is equal to  $\overline{AC}^2$ , and  $4\overline{DE}^2$  to  $\overline{BD}^2$  (P. VIII., C.) : hence,

$$\overline{AB}^2 + \overline{BC}^2 + \overline{CD}^2 + \overline{DA}^2 = \overline{AC}^2 + \overline{BD}^2.$$

That is, the sum of the squares of the sides of a parallelogram, is equal to the sum of the squares of its diagonals.

## PROPOSITION XV. THEOREM.

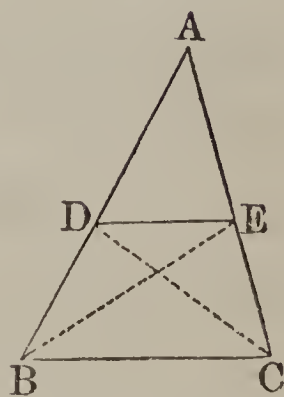
*In any triangle, a line drawn parallel to the base divides the other sides proportionally.*

Let  $ABC$  be a triangle, and  $DE$  a line parallel to the base  $BC$ : then

$$AD : DB :: AE : CE.$$

Draw  $EB$  and  $DC$ . Then, because the triangles  $AED$  and  $DEB$  have their bases in the same line  $AB$ , and their vertices at the same point  $E$ , they will have a common altitude: hence, (P. VI., C.)

$$AED : DEB :: AD : DB.$$



The triangles  $AED$  and  $EDC$ , have their bases in the same line  $AC$ , and their vertices at the same point  $D$ ; they have, therefore, a common altitude; hence,

$$AED : EDC :: AE : EC.$$

But the triangles  $DEB$  and  $EDC$  have a common base  $DE$ , and their vertices in the line  $BC$ , parallel to  $DE$ ; they are, therefore, equal: hence, the two preceding proportions have a couplet in each equal; and consequently, the remaining terms are proportional (B. II., P. IV.), hence,

$$AD : DB :: AE : EC;$$

*which was to be proved.*

*Cor. 1.* We have, by composition (B. II., P. VI.),

$$AD + DB : AD :: AE + EC : AE;$$

or,  $AB : AD :: AC : AE;$

and, in like manner,

$$AB : DB :: AC : EC.$$

*Cor. 2.* If any number of parallels be drawn cutting two lines, they will divide the lines proportionally.

For, let  $O$  be the point where  $AB$  and  $CD$  meet. In the triangle  $OEF$ , the line  $AC$  being parallel to the base  $EF$ , we shall have,

$$OE : AE :: OF : CF.$$

In the triangle  $OGH$ , we shall have,

$$OE : EG :: OF : FH;$$

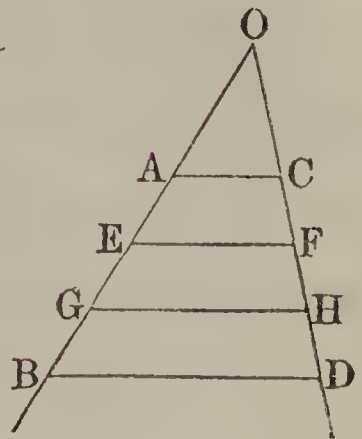
hence (B. II., P. IV., C.),

$$AE : EG :: CF : FH.$$

In like manner,

$$EG : GB :: FH : HD;$$

and so on.



#### PROPOSITION XVI. THEOREM.

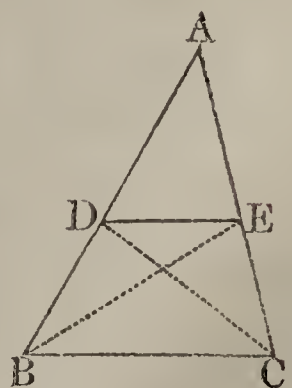
*If a line divides two sides of a triangle proportionally, it will be parallel to the third side.*

Let  $ABC$  be a triangle, and let  $DE$  divide  $AB$  and  $AC$ , so that

$$AD : DB :: AE : EC;$$

then will  $DE$  be parallel to  $BC$ .

Draw  $DC$  and  $EB$ . Then the tri-





angles  $ADE$  and  $DEB$  will have a common altitude ; and consequently, we shall have,

$$ADE : DEB :: AD : DB.$$

The triangles  $ADE$  and  $EDC$  have also a common altitude ; and consequently, we shall have,

$$ADE : EDC :: AE : EC ;$$

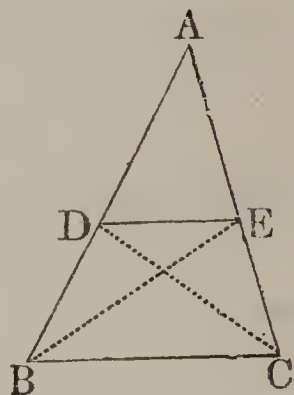
but, by hypothesis,

$$AD : DB :: AE : EC ;$$

hence (B. II., P. IV.),

$$ADE : DEB :: ADE : EDC.$$

The antecedents of this proportion being equal, the consequents will be equal ; that is, the triangles  $DEB$  and  $EDC$  are equal. But these triangles have a common base  $DE$  : hence, their altitudes are equal (P. VI., C.) ; that is, the points  $B$  and  $C$ , of the line  $BC$ , are equally distant from  $DE$ , or  $DE$  prolonged : hence,  $BC$  and  $DE$  are parallel (B. I., P. XXX., C.) ; *which was to be proved.*



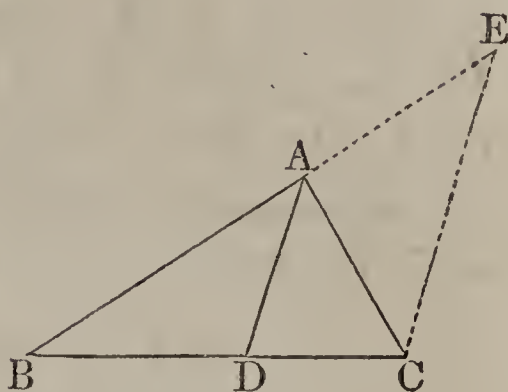
#### PROPOSITION XVII. THEOREM.

*The line which bisects the vertical angle of a triangle, divides the base into segments proportional to the adjacent sides.*

Let  $AD$  bisect the vertical angle  $A$  of the triangle  $BAC$  : then will the segments  $BD$  and  $DC$  be proportional to the adjacent sides  $BA$  and  $CA$ .

From  $C$ , draw  $CE$  parallel to  $DA$ , and produce it

until it meets  $BA$  prolonged, at  $E$ . Then, because  $CE$  and  $DA$  are parallel, the angles  $BAD$  and  $AEC$  are equal (B. I., P. XX., C. 3); the angles  $DAC$  and  $ACE$  are also equal (B. I., P. XX., C. 2). But,  $BAD$  and  $DAC$  are equal, by hypothesis; consequently,  $AEC$  and  $ACE$  are equal: hence, the triangle  $ACE$  is isosceles,  $AE$  being equal to  $AC$ .



In the triangle  $BEC$ , the line  $AD$  is parallel to the base  $EC$ : hence (P. XV.),

$$BA : AE :: BD : DC;$$

or, substituting  $AC$  for its equal  $AE$ ,

$$BA : AC :: BD : DC;$$

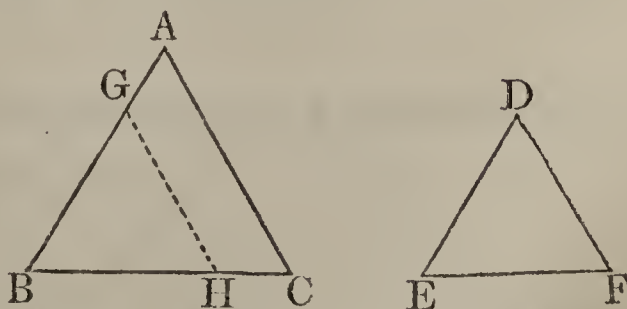
*which was to be proved.*

#### PROPOSITION XVIII. THEOREM.

*Triangles which are mutually equiangular, are similar.*

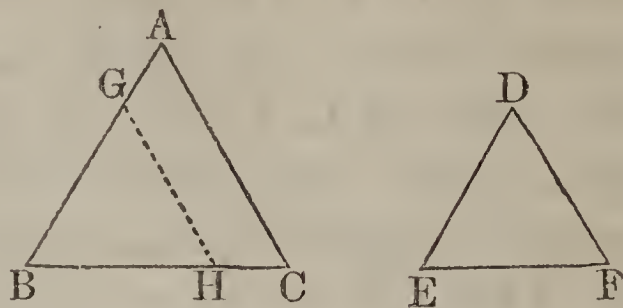
Let the triangles  $ABC$  and  $DEF$  have the angle  $A$  equal to the angle  $D$ , the angle  $B$  to the angle  $E$ , and the angle  $C$  to the angle  $F$ : then will they be similar.

For, place the triangle  $DEF$  upon the triangle  $ABC$ , so that the angle  $E$  shall coincide with the angle  $B$ ; then will the point  $F$  fall at some point  $H$ , of  $BC$ ; the point  $D$  at some point  $G$ , of  $BA$ ;



the side  $DF$  will take the position  $GH$ , and  $BGH$  will be equal to  $EDF$ .

Since the angle  $BHG$  is equal to  $BCA$ ,  $GH$  will be parallel to  $AC$  (B. I., P. XIX., C. 2); and consequently, we shall have (P. XV.),



$$BA : BG :: BC : BH;$$

or, since  $BG$  is equal to  $ED$ , and  $BH$  to  $EF$ ,

$$BA : ED :: BC : EF.$$

In like manner, it may be shown that

$$BC : EF :: CA : FD;$$

and also,

$$CA : FD :: AB : DE;$$

hence, the sides about the equal angles, taken in the same order, are proportional; and consequently, the triangles are similar (D. 1); *which was to be proved.*

*Cor.* If two triangles have two angles in one, equal to two angles in the other, each to each, they will be similar (B. I., P. XXV., C. 2).

#### PROPOSITION XIX. THEOREM.

*Triangles which have their corresponding sides proportional, are similar.*

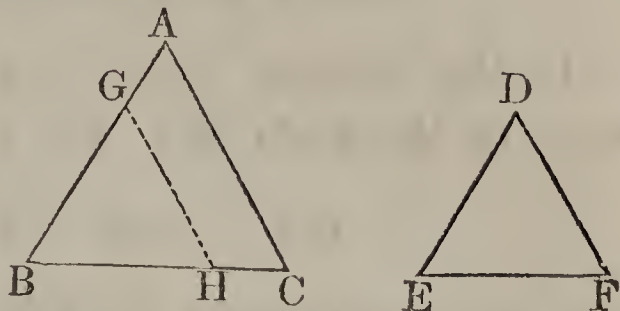
In the triangles  $ABC$  and  $DEF$ , let the corresponding sides be proportional; that is, let



$$AB : DE :: BC : EF : CA : FD ;$$

then will the triangles be similar.

For, on  $BA$  lay off  $BG$  equal to  $ED$ ; on  $BC$  lay off  $BH$  equal to  $EF$ , and draw  $GH$ . Then, because  $BG$  is equal to  $DE$ , and  $BH$  to  $EF$ , we have,



$$BA : BG :: BC : BH ;$$

hence,  $GH$  is parallel to  $AC$  (P. XVI.); and consequently, the triangles  $BAC$  and  $BGH$  are equiangular, and therefore similar: hence,

$$BC : BH :: CA : HG.$$

But, by hypothesis,

$$BC : EF :: CA : FD ;$$

hence (B. II., P. IV., C.), we have,

$$BH : EF :: HG : FD.$$

But,  $BH$  is equal to  $EF$ ; hence,  $HG$  is equal to  $FD$ . The triangles  $BHG$  and  $EFD$  have, therefore, their sides equal, each to each, and consequently, they are equal in all their parts. Now, it has just been shown that  $BHG$  and  $BCA$  are similar: hence,  $EFD$  and  $BCA$  are also similar; which was to be proved.

*Scholium.* In order that polygons may be similar, they must fulfill two conditions: they must be *mutually equiangular*, and *the corresponding sides must be proportional*. In the case of triangles, either of these conditions involves the other, which is not true of any other species of polygons.

## PROPOSITION XX. THEOREM.

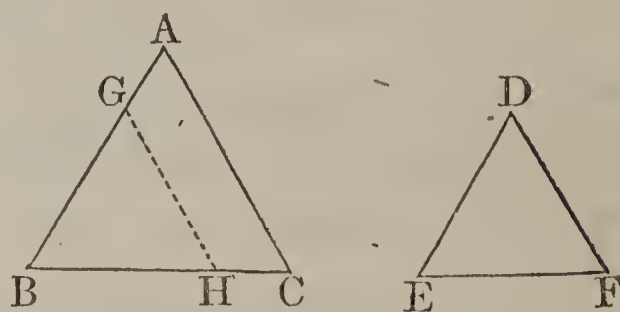
*Triangles which have an angle in each equal, and the including sides proportional, are similar.*

In the triangles  $ABC$  and  $DEF$ , let the angle  $B$  be equal to the angle  $E$ ; and suppose that

$$BA : ED :: BC : EF;$$

then will the triangles be similar.

For, place the angle  $E$  upon its equal  $B$ ;  $F$  will fall at some point of  $BC$ , as  $H$ ;  $D$  will fall at some point of  $BA$ , as



$G$ ;  $DF$  will take the position  $GH$ , and the triangle  $DEF$  will coincide with  $GBH$ , and consequently, will be equal to it.

But, from the assumed proportion, and because  $BG$  is equal to  $ED$ , and  $BH$  to  $EF$  we have,

$$BA : BG :: BC : BH;$$

hence,  $GH$  is parallel to  $AC$ ; and consequently,  $BAC$  and  $BGH$  are equiangular, and therefore similar. But,  $EDF$  is equal to  $BGH$ : hence, it is also similar to  $BAC$ ; which was to be proved.

## PROPOSITION XXI. THEOREM.

*Triangles which have their sides parallel, each to each, or perpendicular, each to each, are similar.*

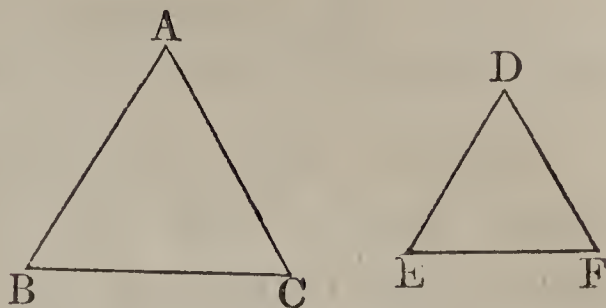
1°. Let the triangles  $ABC$  and  $DEF$  have the side  $AB$  parallel to  $DE$ ,  $BC$  to  $EF$ , and  $CA$  to  $FD$ : then will they be similar.

For, since the side  $AB$  is parallel to  $DE$ , and  $BC$  to  $EF$ , the angle  $B$  is equal to the angle  $E$  (B. I., P. XXIV.) ; in like manner,

the angle  $C$  is equal to the angle  $F$ , and the angle  $A$  to the angle  $D$  ;

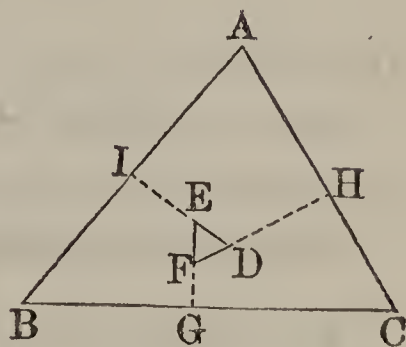
the triangles are, therefore, mutually equiangular, and

consequently, are similar (P. XVIII.) ; *which was to be proved.*



2°. Let the triangles  $ABC$  and  $DEF$  have the side  $AB$  perpendicular to  $DE$ ,  $BC$  to  $EF$ , and  $CA$  to  $FD$  : then will they be similar.

For, prolong the sides of the triangle  $DEF$  till they meet the sides of the triangle  $ABC$ . The sum of the interior angles of the quadrilateral  $BIEG$  is equal to four right angles (B. I., P. XXVI.) ; but, the angles  $EIB$  and  $EGB$  are each right



angles, by hypothesis ; hence, the sum of the angles  $IEG$   $IBG$  is equal to two right angles ; the sum of the angles  $IEG$  and  $DEF$  is equal to two right angles, because they are adjacent ; and since things which are equal to the same thing are equal to each other, the sum of the angles  $IEG$  and  $IBG$  is equal to the sum of the angles  $IEG$  and  $DEF$  ; or, taking away the common part  $IEG$ , we have the angle  $IBG$  equal to the angle  $DEF$ . In like manner, the angle  $GCH$  may be proved equal to the angle  $EFD$ , and the angle  $HAI$  to the angle  $EDF$  ; the triangles  $ABC$  and  $DEF$  are, therefore, mutually equiangular, and consequently, similar ; *which was to be proved.*

*Cor. 1.* In the first case, the parallel sides are homolo-



gous; in the second case, the perpendicular sides are homologous.

*Cor. 2.* The homologous angles are those included by sides respectively parallel or perpendicular to each other.

*Scholium.* When two triangles have their sides perpendicular, each to each, they may have a different relative position from that shown in the figure. But we can always construct a triangle within the triangle  $ABC$ , whose sides shall be parallel to those of the other triangle, and then the demonstration will be the same as above.

PROPOSITION XXII. THEOREM.

*If a line be drawn parallel to the base of a triangle, and lines be drawn from the vertex of the triangle to points of the base, these lines will divide the base and the parallel proportionally.*

Let  $ABC$  be a triangle,  $BC$  its base,  $A$  its vertex,  $DE$  parallel to  $BC$ , and  $AF$ ,  $AG$ ,  $AH$ , lines drawn from  $A$  to points of the base: then will

$$DI : BF :: IK : FG :: KL : GH :: LE : HC.$$

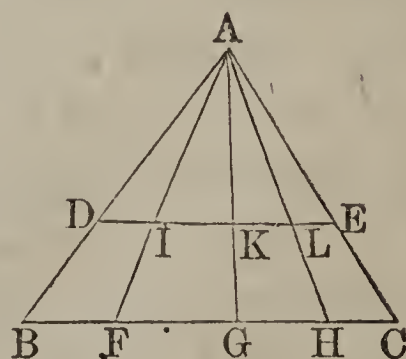
For, the triangles  $AID$  and  $AFB$ , being similar (P. XXI.), we have,

$$AI : AF :: DI : BF;$$

and, the triangles  $AIK$  and  $AFG$ , being similar, we have,

$$AI : AF :: IK : FG;$$

hence, (B. II., P. IV.), we have,



$$DI : BF :: IK : FG.$$

In like manner,

$$IK : FG :: KL : GH,$$

and,

$$KL : GH :: LE : HC ;$$

hence (B. II., P. IV.),

$$DI : BF :: IK : FG :: KL : GH :: LE : HC ;$$

which was to be proved.

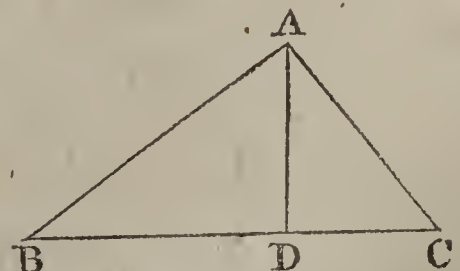
*Cor.* If  $BC$  is divided into equal parts at  $F$ ,  $G$ , and  $H$ , then will  $DE$  be divided into equal parts, at  $I$ ,  $K$ , and  $L$ .

#### PROPOSITION XXIII. THEOREM.

*If, in a right-angled triangle, a perpendicular be drawn from the vertex of the right angle to the hypotenuse:*

- 1°. *The triangles on each side of the perpendicular will be similar to the given triangle, and to each other:*
- 2°. *Each side about the right angle will be a mean proportional between the hypotenuse and the adjacent segment:*
- 3°. *The perpendicular will be a mean proportional between the two segments of the hypotenuse.*

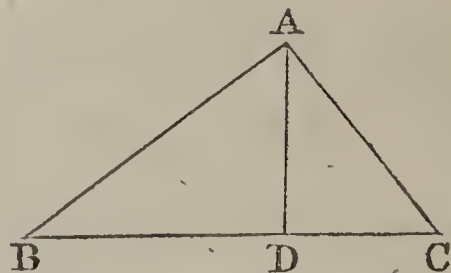
1°. Let  $ABC$  be a right-angled triangle,  $A$  the vertex of the right angle,  $BC$  the hypotenuse, and  $AD$  perpendicular to  $BC$ : then will  $ADB$  and  $ADC$  be similar to  $ABC$ , and consequently, similar to each other.



The triangles  $ADB$  and  $ABC$  have the angle  $B$  common, and the angles  $ADB$  and

$BAC$  equal, because both are right angles; they are, therefore, similar (P. XVIII., C). In like manner, it may be shown that the triangles  $ADC$  and  $ABC$  are similar; and since  $ADB$  and  $ADC$  are both similar to  $ABC$ , they are similar to each other; *which was to be proved.*

2°.  $AB$  will be a mean proportional between  $BC$  and  $BD$ ; and  $AC$  will be a mean proportional between  $CB$  and  $CD$ .



For, the triangles  $ADB$  and  $BAC$  being similar, their homologous sides are proportional: hence,

$$BC : AB :: AB : BD.$$

In like manner,

$$BC : AC :: AC : DC;$$

*which was to be proved.*

3°.  $AD$  will be a mean proportional between  $BD$  and  $DC$ . For, the triangles  $ADB$  and  $ADC$  being similar, their homologous sides are proportional; hence,

$$BD : AD :: AD : DC;$$

*which was to be proved.*

Cor. 1. From the proportions,

$$BC : AB :: AB : BD,$$

and,

$$BC : AC :: AC : DC,$$

we have (B. II., P. I.),

$$\overline{AB}^2 = BC \times BD,$$

and,

$$\overline{AC}^2 = BC \times DC;$$



whence, by addition,

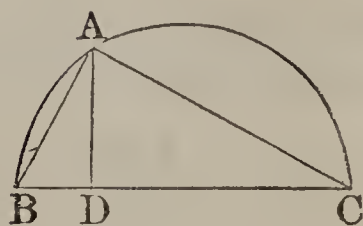
$$\overline{AB}^2 + \overline{AC}^2 = BC(BD + DC) ;$$

or,

$$\overline{AB}^2 + \overline{AC}^2 = \overline{BC}^2 ;$$

as was shown in Proposition XI.

*Cor. 2.* If from any point  $A$ , in a semi-circumference  $BAC$ , chords be drawn to the extremities  $B$  and  $C$  of the diameter  $BC$ , and a perpendicular  $AD$  be drawn to the diameter: then will  $ABC$  be a right-angled triangle, right-angled at  $A$ ; and from what was proved above, each chord will be a mean proportional between the diameter and the adjacent segment; and, the perpendicular will be a mean proportional between the segments of the diameter.

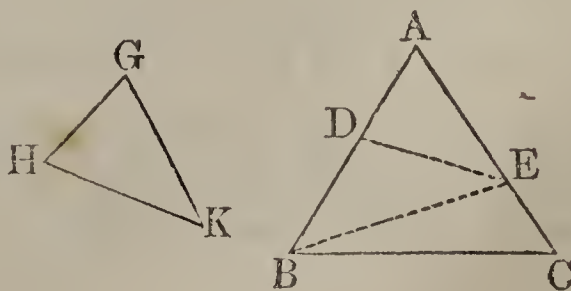


#### PROPOSITION XXIV. THEOREM.

*Triangles which have an angle in each equal, are to each other as the rectangles of the including sides.*

Let the triangles  $GHK$  and  $ABC$  have the angles  $G$  and  $A$  equal: then will they be to each other as the rectangles of the sides about these angles.

For, lay off  $AD$  equal to  $GH$ ,  $AE$  to  $GK$ , and draw  $DE$ ; then will the triangles  $ADE$  and  $GHK$  be equal in all their parts. Draw  $EB$ .

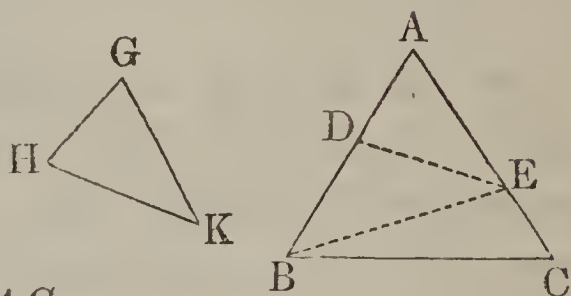


The triangles  $ADE$  and  $ABE$  have their bases in the same line  $AB$ , and a common vertex  $E$ ; therefore, they have the same altitude, and consequently, are to each other as their bases; that is,

$$ADE : ABE :: AD : AB.$$

The triangles  $ABE$  and  $ABC$ , have their bases in the same line  $AC$ , and a common vertex  $B$ ; hence,

$$ABE : ABC :: AE : AC;$$



multiplying these proportions, term by term, and omitting the common factor  $ABE$  (B. II., P. VII.), we have,

$$ADE : ABC :: AD \times AE : AB \times AC;$$

substituting for  $ADE$ , its equal,  $GHK$ , and for  $AD \times AE$ , its equal,  $GH \times GK$ , we have,

$$GHK : ABC :: GH \times GK : AB \times AC;$$

*which was to be proved.*

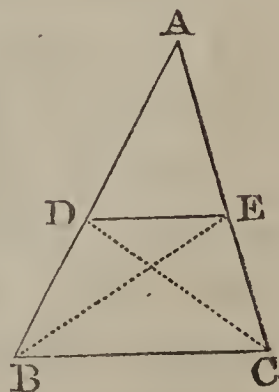
*Cor.* If  $ADE$  and  $ABC$  are similar, the angles  $D$  and  $B$  being homologous,  $DE$  will be parallel to  $BC$ , and we shall have,

$$AD : AB :: AE : AC;$$

hence (B. II., P. IV.), we have,

$$ADE : ABE :: ABE : ABC;$$

that is,  $ABE$  is a mean proportional between  $ADE$  and  $ABC$ .



## PROPOSITION XXV. THEOREM.

*Similar triangles are to each other as the squares of their homologous sides.*

Let the triangles  $ABC$  and  $DEF$  be similar, the angle  $A$  being equal to the angle  $D$ ,  $B$  to  $E$ , and  $C$  to  $F$ : then will the triangles be to each other as the squares of any two homologous sides.

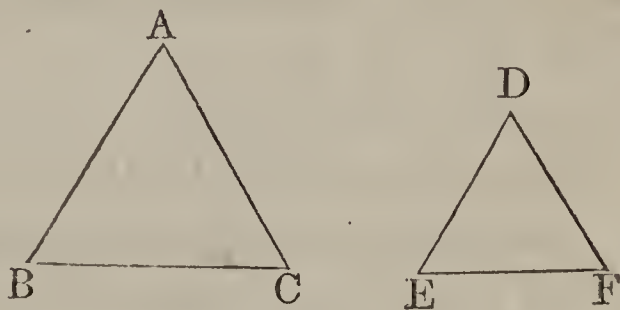
Because the angles  $A$  and  $D$  are equal, we have (P. XXIV.),

$$ABC : DEF :: AB \times AC : DE \times DF;$$

and, because the triangles are similar, we have,

$$AB : DE :: AC : DF;$$

multiplying the terms of this proportion by the corresponding terms of the proportion,



$$AC : DF :: AC : DF,$$

we have (B. II., P. XII.),

$$AB \times AC : DE \times DF :: \overline{AC}^2 : \overline{DF}^2;$$

combining this, with the first proportion (B. II., P. IV.), we have,

$$ABC : DEF :: \overline{AC}^2 : \overline{DF}^2.$$

In like manner, it may be shown that the triangles are to each other as the squares of  $AB$  and  $DE$ , or of  $BC$  and  $EF$ ; which was to be proved.

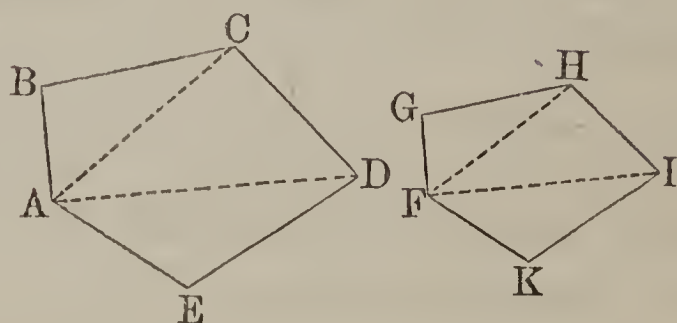


## PROPOSITION XXVI. THEOREM.

*Similar polygons may be divided into the same number of triangles, similar, each to each, and similarly placed.*

Let  $ABCDE$  and  $FGHIK$  be two similar polygons, the angle  $A$  being equal to the angle  $F$ ,  $B$  to  $G$ ,  $C$  to  $H$ , and so on: then can they be divided into the same number of similar triangles, similarly placed.

For, from  $A$  draw the diagonals  $AC$ ,  $AD$ , and from  $F$ , homologous with  $A$ , draw the diagonals  $FH$ ,  $FI$ , to the vertices  $H$  and  $I$ , homologous with  $C$  and  $D$ .



Because the polygons are similar, the triangles  $ABC$  and  $FGH$  have the angles  $B$  and  $G$  equal, and the sides about these angles proportional; they are, therefore, similar (P. XX.). Since these triangles are similar, we have the angle  $ACB$  equal to  $FHG$ , and the sides  $AC$  and  $FH$ , proportional to  $BC$  and  $GH$ , or to  $CD$  and  $HI$ . The angle  $BCD$  being equal to the angle  $GHI$ , if we take from the first the angle  $ACB$ , and from the second the equal angle  $FHG$ , we shall have the angle  $ACD$  equal to the angle  $FHI$ : hence, the triangles  $ACD$  and  $FHI$  have an angle in each equal, and the including sides proportional; they are therefore similar.

In like manner, it may be shown that  $ADE$  and  $FIK$  are similar; which was to be proved.

*Cor. 1.* The corresponding triangles in the two polygons are homologous triangles, and the corresponding diagonals are homologous diagonals.

*Cor. 2.* Any two homologous triangles are *like parts* of the polygons to which they belong.

For,  $ABC$  and  $FGH$  being similar, we have,

$$ABC : FGH :: \overline{AC}^2 : \overline{FH}^2 ;$$

and, for a like reason,

$$ACD : FHI :: \overline{AC}^2 : \overline{FH}^2 ;$$

whence,

$$ABC : FGH :: ACD : FHI ;$$

and, in like manner,

$$ACD : FHI :: ADE : IKF.$$

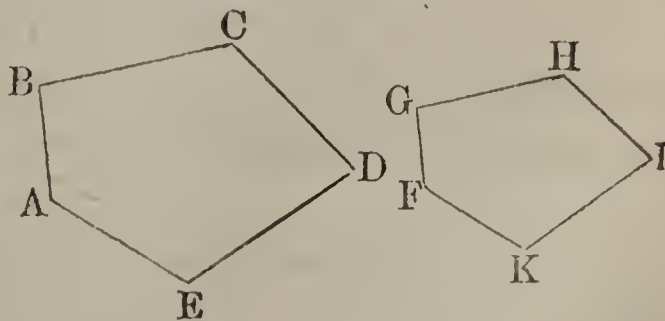
*Cor. 3.* If two polygons are made up of similar triangles, similarly placed, the polygons themselves will be similar.

#### PROPOSITION XXVII. THEOREM.

*The perimeters of similar polygons are to each other as any two homologous sides ; and the polygons are to each other as the squares of any two homologous sides.*

1°. Let  $ABCDE$  and  $FGHIK$  be similar polygons : then will their perimeters be to each other as any two homologous sides.

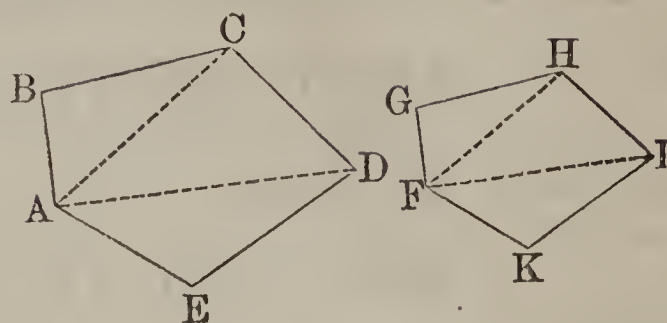
For, any two homologous sides, as,  $AB$  and  $FG$ , are like parts of the perimeters to which they belong : hence (B. II., P. IX.), the perimeters of the



polygons are to each other as  $AB$  to  $FG$ , or as any other two homologous sides ; *which was to be proved.*

2°. The polygons will be to each other as the squares of any two homologous sides.

For, let the polygons be divided into homologous triangles (P. XXVI., C. 1); then, because the homologous triangles  $ABC$  and  $FGH$  are



like parts of the polygons to which they belong, the polygons will be to each other as these triangles; but these triangles, being similar, are to each other as the squares of  $AB$  and  $FG$ : hence, the polygons are to each other as the squares of  $AB$  and  $FG$ , or as the squares of any other two homologous sides; *which was to be proved.*

*Cor.* 1. Perimeters of similar polygons are to each other as their homologous diagonals, or as any other homologous lines; and the polygons are to each other as the squares of their homologous diagonals, or as the squares of any other homologous lines.

*Cor.* 2. If the three sides of a right-angled triangle be made homologous sides of three similar polygons, these polygons will be to each other as the squares of the sides of the triangle. But the square of the hypotenuse is equal to the sum of the squares of the other sides, and consequently, *the polygon on the hypotenuse will be equal to the sum of the polygons on the other sides.*

#### PROPOSITION XXVIII. THEOREM.

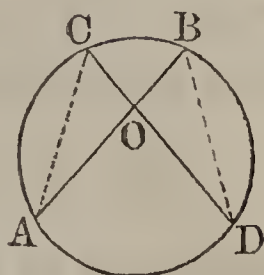
*If two chords intersect in a circle, their segments will be reciprocally proportional.*

Let the chords  $AB$  and  $CD$  intersect at  $O$ : then



will their segments be reciprocally proportional; that is, one segment of the first will be to one segment of the second, as the remaining segment of the second is to the remaining segment of the first.

For, draw  $CD$  and  $BA$ . Then will the angles  $ODB$  and  $OAC$  be equal, because each is measured by half of the arc  $CB$  (B. III., P. XVIII.). The angles  $OBD$  and  $OCA$ , will also be equal, because each is measured by half of the arc  $AD$ : hence, the triangles  $OBD$  and  $OCA$  are similar (P. XIX., C.), and consequently, their homologous sides are proportional: hence,



$$DO : AO :: OB : OC ;$$

*which was to be proved.*

*Cor.* From the above proportion, we have,

$$DO \times OC = AO \times OB ;$$

that is, *the rectangle of the segments of one chord is equal to the rectangle of the segments of the other.*

#### PROPOSITION XXIX. THEOREM.

*If from a point without a circle, two secants be drawn terminating in the concave arc, they will be reciprocally proportional to their external segments.*

Let  $OB$  and  $OC$  be two secants terminating in the concave arc of the circle  $BCD$ : then will

$$OB : OC :: OD : OA.$$

For, draw  $AC$  and  $DB$ . The triangles  $ODB$  and  $OAC$  have the angle  $O$  common, and the angles  $OBD$  and  $OCA$  equal, because each is measured by half of the arc  $AD$ : hence, they are similar, and consequently, their homologous sides are proportional; whence,

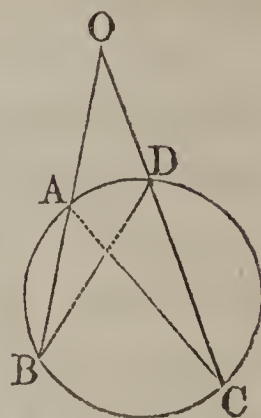
$$OB : OC :: OD : OA ;$$

*which was to be proved.*

*Cor.* From the above proportion, we have,

$$OB \times OA = OC \times OD ;$$

that is, *the rectangles of each secant and its external segment are equal.*



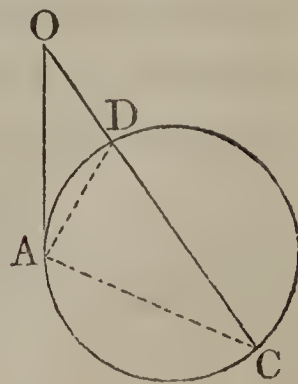
#### PROPOSITION XXX. THEOREM.

*If from a point without a circle, a tangent and a secant be drawn, the secant terminating in the concave arc, the tangent will be a mean proportional between the secant and its external segment.*

Let  $ADC$  be a circle,  $OC$  a secant, and  $OA$  a tangent: then will

$$OC : OA :: OA : OD.$$

For, draw  $AD$  and  $AC$ . The triangles  $OAD$  and  $OAC$  will have the angle  $O$  common, and the angles  $OAD$  and  $ACD$  equal, because each is measured by half of the arc  $AD$  (B. III., P. XVIII., P. XXI.); the triangles are therefore similar, and consequently, their



homologous sides are proportional: hence,

$$OC : OA :: OA : OD ;$$

which was to be proved.

Cor. From the above proportion, we have,

$$\overline{AO}^2 = OC \times OD ;$$

that is, *the square of the tangent is equal to the rectangle of the secant and its external segment.*

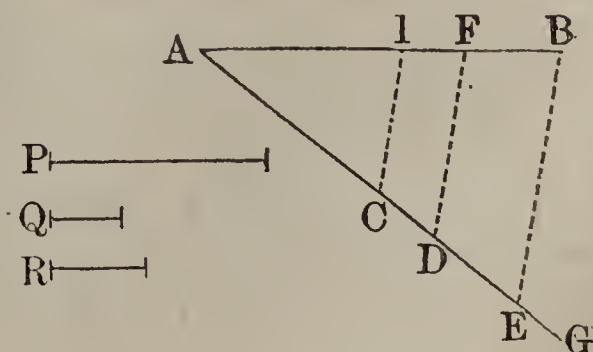
## PRACTICAL APPLICATIONS.

### PROBLEM I.

*To divide a given line into parts proportional to given lines, also into equal parts.*

1°. Let  $AB$  be a given line, and let it be required to divide it into parts proportional to the lines  $P$ ,  $Q$  and  $R$ .

From one extremity  $A$ , draw the indefinite line  $AG$ , making any angle with  $AB$ ; lay off  $AC$  equal to  $P$ ,  $CD$  equal to  $Q$ , and  $DE$  equal to  $R$ ; draw  $EB$ , and from the points  $C$  and  $D$ ,

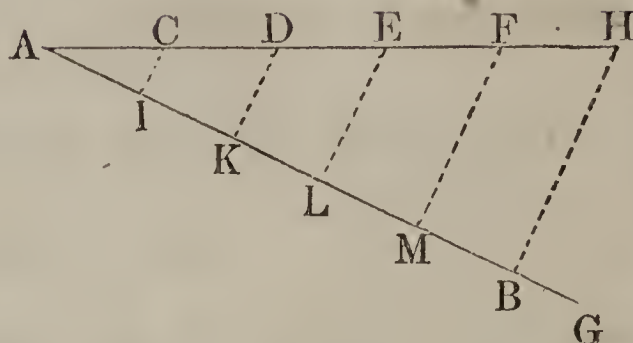


draw  $CI$  and  $DF$  parallel to  $EB$ : then will  $AI$ ,  $IF$ , and  $FB$ , be proportional to  $P$ ,  $Q$ , and  $R$  (P XV., C. 2).



2°. Let  $AH$  be a given line, and let it be required to divide it into any number of equal parts, say five.

From one extremity  $A$ , draw the indefinite line  $AG$ ; take  $AI$  equal to any convenient line, and lay off  $IK$ ,  $KL$ ,  $LM$ , and  $MB$ , each equal to  $AI$ . Draw

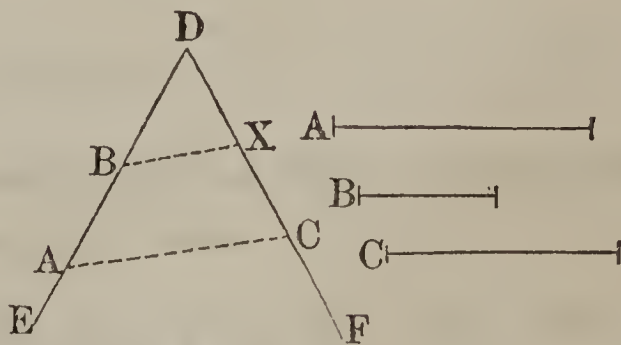


$BH$ , and from  $I$ ,  $K$ ,  $L$ , and  $M$ , draw the lines  $IC$ ,  $KD$ ,  $LE$ , and  $MF$ , parallel to  $BH$ : then will  $AH$  be divided into equal parts at  $C$ ,  $D$ ,  $E$ , and  $F$  (P. XV., C. 2).

#### PROBLEM II.

*To construct a fourth proportional to three given lines.*

Let  $A$ ,  $B$ , and  $C$ , be the given lines. Draw  $DE$  and  $DF$ , making any convenient angle with each other. Lay off  $DA$  equal to  $A$ ,  $DB$  equal to  $B$ , and  $DC$  equal



to  $C$ ; draw  $AC$ , and from  $B$  draw  $BX$  parallel to  $AC$ : then will  $DX$  be the fourth proportional required.

For (P. XV., C.), we have,

$$DA : DB :: DC : DX;$$

or,

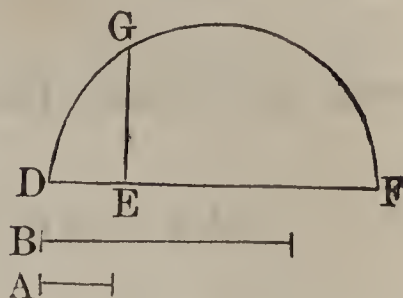
$$A : B :: C : DX.$$

*Cor.* If  $DC$  is made equal to  $DB$ ,  $DX$  will be a third proportional to  $DA$  and  $DB$ , or to  $A$  and  $B$ .

## PROBLEM III.

*To construct a mean proportional between two given lines.*

Let  $A$  and  $B$  be the given lines. On an indefinite line, lay off  $DE$  equal to  $A$ , and  $EF$  equal to  $B$ ; on  $DF$  as a diameter describe the semi-circle  $DGF$ , and draw  $EG$  perpendicular to  $DF$ : then will  $EG$  be the mean proportional required.



For (P. XXIII., C. 2), we have,

$$DE : EG :: EG : EF;$$

or,

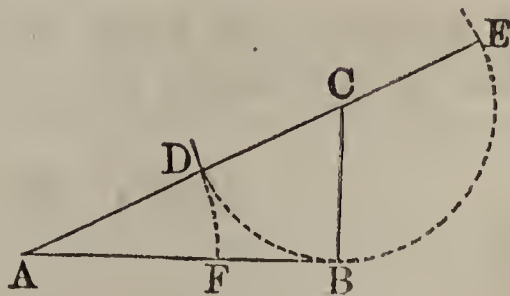
$$A : EG :: EG : B.$$

## PROBLEM IV.

*To divide a given line into two such parts, that the greater part shall be a mean proportional between the whole line and the other part.*

Let  $AB$  be the given line.

At the extremity  $B$ , draw  $BC$  perpendicular to  $AB$ , and make it equal to half of  $AB$ . With  $C$  as a centre, and  $CB$  as a radius, describe the arc  $DBE$ ; draw  $AC$ , and produce it till it terminates in the concave arc at  $E$ ; with  $A$  as centre and  $AD$  as radius, describe the arc  $DF$ : then will  $AF$  be the greater part required.







## PROBLEM V.

*Through a given point, in a given angle, to draw a line so that the segments between the point and the sides of the angle shall be equal.*

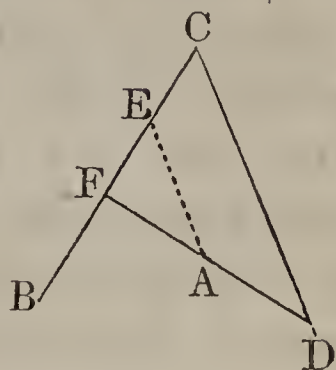
Let  $BCD$  be the given angle, and  $A$  the given point.

Through  $A$ , draw  $AE$  parallel to  $DC$ ; lay off  $EF$  equal to  $CE$ , and draw  $FAD$ : then will  $AF$  and  $AD$  be the segments required.

For (P. XV.), we have,

$$FA : AD :: FE : EC ;$$

but,  $FE$  is equal to  $EC$ ; hence,  $FA$  is equal to  $AD$ .

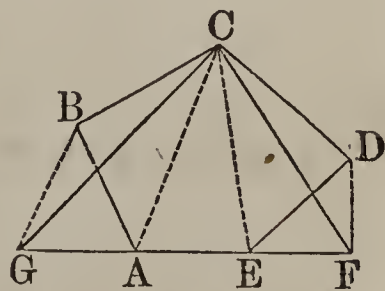


## PROBLEM VI.

*To construct a triangle equal to a given polygon.*

Let  $ABCDE$  be the given polygon.

Draw  $CA$ ; produce  $EA$ , and draw  $BG$  parallel to  $CA$ ; draw the line  $CG$ . Then the triangles  $BAC$  and  $GAC$  have the common base  $AC$ , and because their vertices  $B$  and  $G$  lie in the



same line  $BG$  parallel to the base, their altitudes are equal, and consequently, the triangles are equal: hence, the polygon  $GCDE$  is equal to the polygon  $ABCDE$ .

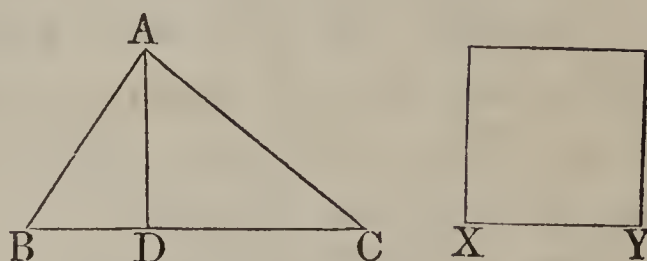
Again, draw  $CE$ ; produce  $AE$  and draw  $DF$  parallel to  $CE$ ; draw also  $CF$ ; then will the triangles  $FCE$  and  $DCE$  be equal: hence, the triangle  $GCF$  is equal to the polygon  $GCDE$ , and consequently, to the given polygon. In like manner, a triangle may be constructed equal to any other given polygon.

## PROBLEM VII.

*To construct a square equal to a given triangle.*

Let  $ABC$  be the given triangle,  $AD$  its altitude, and  $BC$  its base.

Construct a mean proportional between  $AD$  and half of  $BC$  (Prob. III.). Let  $XY$  be that mean proportional, and on it, as a side, construct a square: then will this be the square required. For, from the construction,



$$\overline{XY}^2 = \frac{1}{2}BC \times AD = \text{area } ABC.$$

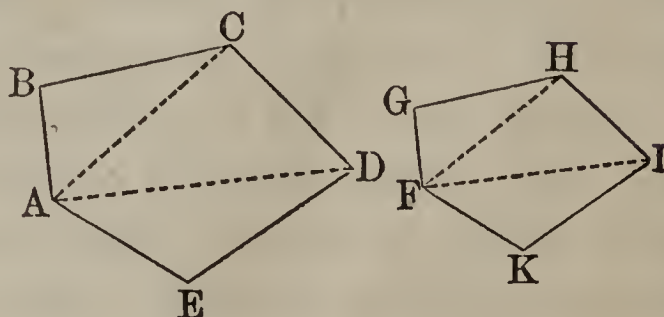
*Scholium.* By means of Problems VI. and VII., a square may be constructed equal to any given polygon.

## PROBLEM VIII.

*On a given line, to construct a polygon similar to a given polygon.*

Let  $FG$  be the given line, and  $ABCDE$  the given polygon. Draw  $AC$  and  $AD$ .

At  $F$ , construct the angle  $GFH$  equal to  $BAC$ , and at  $G$  the angle  $FGH$  equal to  $ABC$ ; then will  $FGH$  be similar to  $ABC$  (P. XVIII., C.)



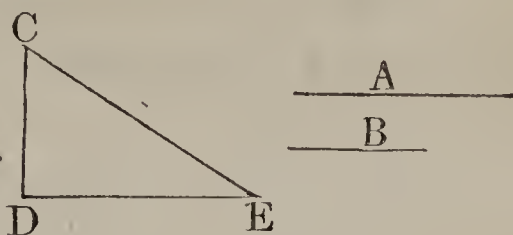
In like manner, construct the triangle  $FHI$  similar to  $ACD$ , and  $FIK$  similar to  $ADE$ ; then will the polygon  $FGHIK$  be similar to the polygon  $ABCDE$  (P. XXVI., C.).

## PROBLEM IX.

*To construct a square equal to the sum of two given squares, also a square equal to the difference of two given squares.*

1°. Let  $A$  and  $B$  be the sides of the given squares, and let  $A$  be the greater.

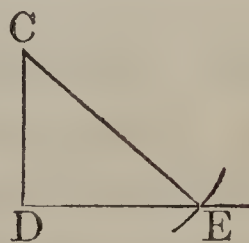
Construct a right angle  $CDE$ ; make  $DE$  equal to  $A$ , and  $DC$  equal to  $B$ ; draw  $CE$ , and on it



construct a square: this square will be equal to the sum of the given squares (P. XI.).

2°. Construct a right angle  $CDE$ .

Lay off  $DC$  equal to  $B$ ; with  $C$  as a centre, and  $CE$ , equal to  $A$ , as a radius, describe an arc cutting  $DE$  at  $E$ ; draw  $CE$ , and on  $DE$  construct a square: this square will be equal to the difference of the given squares (P. XI., C. 1).



*Scholium.* By means of Probs. VI., VII., VIII., and IX., a polygon may be constructed similar to either of two given similar polygons, and equal to their sum, or to their difference.



## BOOK V.

### REGULAR POLYGONS.—AREA OF THE CIRCLE.

#### DEFINITION.

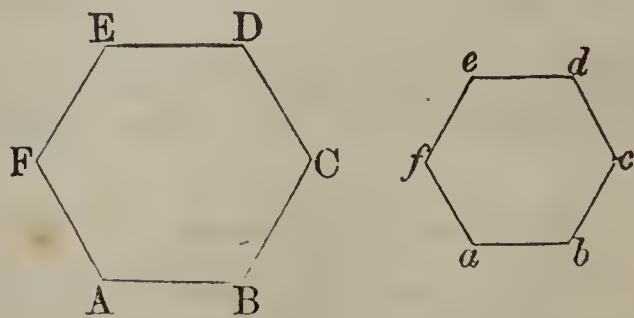
1. A REGULAR POLYGON is a polygon which is both equilateral and equiangular.

#### PROPOSITION I. THEOREM.

*Regular polygons of the same number of sides are similar.*

Let  $ABCDEF$  and  $abcdef$  be regular polygons of the same number of sides: then will they be similar.

For, the corresponding angles in each are equal, because any angle in either polygon is equal to twice as many right angles as the polygon has sides, less four, di-



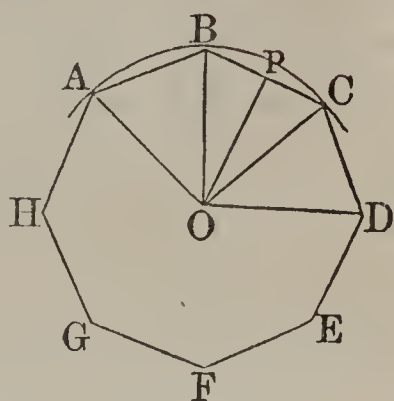
vided by the number of angles (B. I., P. XXVI., C. 4); and further, the corresponding sides are proportional, because all the sides of either polygon are equal (D. 1): hence, the polygons are similar (B. IV., D. 1); *which was to be proved.*

## PROPOSITION II. THEOREM.

*The circumference of a circle may be circumscribed about any regular polygon ; a circle may also be inscribed in it.*

1°. Let  $ABCF$  be a regular polygon : then can the circumference of a circle be circumscribed about it.

For, through three consecutive vertices  $A, B, C$ , describe the circumference of a circle (B. III., Problem XIII., S.). Its centre  $O$  will lie on  $PO$ , drawn perpendicular to  $BC$ , at its middle point  $P$ ; draw  $OA$  and  $OD$ .



Let the quadrilateral  $OPCD$  be turned about the line  $OP$ , until  $PC$  falls on  $PB$ ; then, because the angle  $C$  is equal to  $B$ , the side  $CD$  will take the direction  $BA$ ; and because  $CD$  is equal to  $BA$ , the vertex  $D$ , will fall upon the vertex  $A$ ; and consequently, the line  $OD$  will coincide with  $OA$ , and is, therefore, equal to it: hence, the circumference which passes through  $A, B$ , and  $C$ , will pass through  $D$ . In like manner, it may be shown that it will pass through all of the other vertices: hence, it is circumscribed about the polygon; *which was to be proved.*

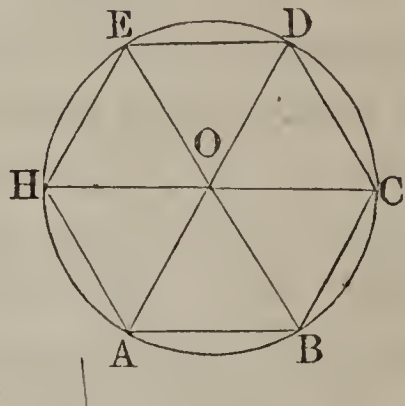
2°. A circle may be inscribed in the polygon.

For, the sides  $AB, BC$ , &c., being equal chords of the circumscribed circle, are equidistant from the centre  $O$ : hence, if a circle be described from  $O$  as a centre, with  $OP$  as a radius, it will be tangent to all of the sides of the polygon, and consequently, will be inscribed in it; *which was to be proved.*

*Scholium.* If the circumference of a circle be divided into equal arcs, the chords of these arcs will be sides of a regular inscribed polygon.

For, the sides are equal, because they are chords of equal arcs, and the angles are equal, because they are measured by halves of equal arcs.

If the vertices  $A, B, C, \&c.$ , of a regular inscribed polygon be joined with the centre  $O$ , the triangles thus formed will be equal, because their sides are equal, each to each: hence, all of the angles about the point  $O$  are equal to each other.



#### DEFINITIONS.

1. The CENTRE OF A REGULAR POLYGON, is the common centre of the circumscribed and inscribed circles.

2. The ANGLE AT THE CENTRE, is the angle formed by drawing lines from the centre to the extremities of either side.

The angle at the centre is equal to four right angles divided by the number of sides of the polygon.

3. The APOTHEM, is the distance from the centre to either side.

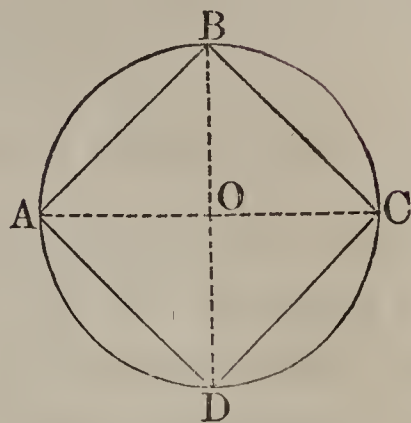
The apothem is equal to the radius of the inscribed circle.



## PROPOSITION III. PROBLEM.

*To inscribe a square in a given circle.*

Let  $ABCD$  be the given circle. Draw any two diameters  $AC$  and  $BD$  perpendicular to each other; they will divide the circumference into four equal arcs (B. III., P. XVII., S.). Draw the chords  $AB$ ,  $BC$ ,  $CD$ , and  $DA$ : then will the figure  $ABCD$  be the square required (P. II., S.).



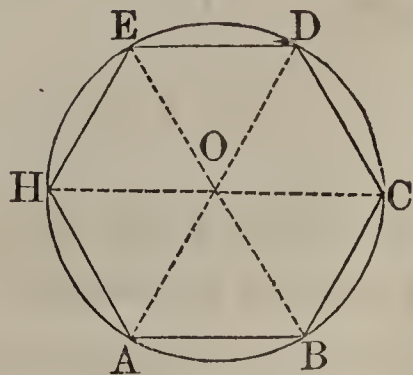
*Scholium.* The radius is to the side of the inscribed square as 1 is to  $\sqrt{2}$ .

## PROPOSITION IV. THEOREM.

*If a regular hexagon be inscribed in a circle, any side will be equal to the radius of the circle.*

Let  $ABD$  be a circle, and  $ABCDEH$  a regular inscribed hexagon: then will any side, as  $AB$ , be equal to the radius of the circle.

Draw the radii  $OA$  and  $OB$ . Then will the angle  $AOB$  be equal to one-sixth of four right angles, or to two-thirds of one right angle, because it is an angle at the centre (P. II., D. 2). The sum of the two angles  $OAB$  and  $OBA$  is, consequently, equal to four-thirds of a right angle (B. I., P. XXV., C. 1); but, the angles  $OAB$  and  $OBA$  are equal, because the opposite sides  $OB$  and  $OA$  are equal: hence, each is equal to



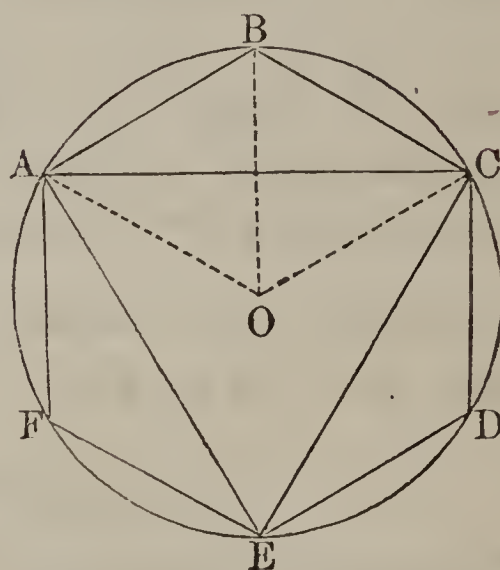
two-thirds of a right angle. The three angles of the triangle  $AOB$  are therefore, equal, and consequently, the triangle is equilateral: hence,  $AB$  is equal to  $OA$ ; which was to be proved.

PROPOSITION V. PROBLEM.

*To inscribe a regular hexagon in a given circle.*

Let  $ABE$  be a circle, and  $O$  its centre.

Beginning at any point of the circumference, as  $A$ , apply the radius  $OA$  six times as a chord; then will  $ABCDEF$  be the hexagon required (P. IV.).



*Cor. 1.* If the alternate vertices of the regular hexagon be joined by the lines  $AC$ ,  $CE$ , and  $EA$ , the inscribed triangle  $ACE$  will be equilateral (P. II., S.).

*Cor. 2.* If we draw the radii  $OA$  and  $OC$ , the figure  $AOCB$  will be a rhombus, because its sides are equal: hence (B. IV., P. XIV., C.), we have,

$$\overline{AB}^2 + \overline{BC}^2 + \overline{OA}^2 + \overline{OC}^2 = \overline{AC}^2 + \overline{OB}^2;$$

or, taking away from the first member the quantity  $\overline{OA}^2$ , and from the second its equal  $\overline{OB}^2$ , and reducing, we have,

$$3\overline{OA}^2 = \overline{AC}^2;$$

whence (B. II., P. II.),

$$\overline{AC}^2 : \overline{OA}^2 :: 3 : 1;$$

or (B. II., P. XII., C. 2),

$$AC : OA :: \sqrt{3} : 1;$$

that is, *the side of an inscribed equilateral triangle is to the radius, as the square root of 3 is to 1.*

PROPOSITION VI. THEOREM.

*If the radius of a circle be divided in extreme and mean ratio, the greater segment will be equal to one side of a regular inscribed decagon.*

Let  $ACG$  be a circle,  $OA$  its radius, and  $AB$ , equal to  $OM$ , the greater segment of  $OA$  when divided in extreme and mean ratio: then will  $AB$  be equal to the side of a regular inscribed decagon.

Draw  $OB$  and  $BM$ . We have, by hypothesis,

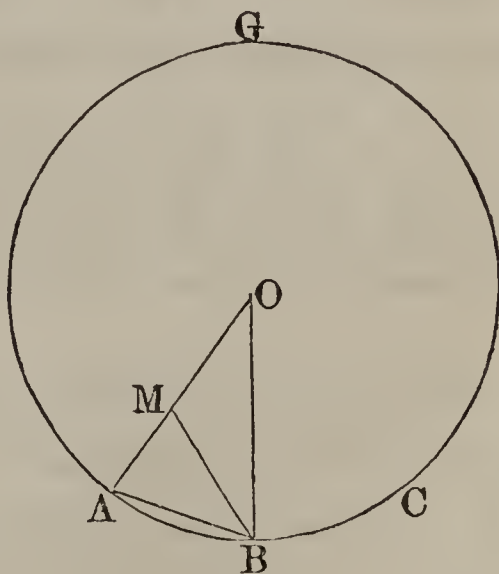
$$AO : OM :: OM : AM;$$

or, since  $AB$  is equal to  $OM$ , we have,

$$AO : AB :: AB : AM;$$

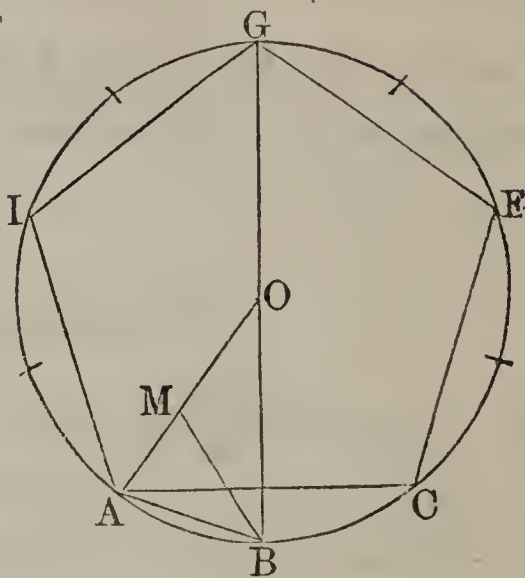
hence, the triangles  $OAB$  and  $BAM$  have the sides about their common angle

$BAM$ , proportional; they are, therefore, similar (B. IV., P. XX.). But, the triangle  $OAB$  is isosceles; hence,  $BAM$  is also isosceles, and consequently, the side  $BM$  is equal to  $AB$ . But,  $AB$  is equal to  $OM$ , by hypothesis: hence,  $BM$  is equal to  $OM$ , and consequently, the angles  $MOB$





and  $\angle MBO$  are equal. The angle  $\angle AMB$  being an exterior angle of the triangle  $OMB$ , is equal to the sum of the angles  $\angle MOB$  and  $\angle MBO$ , or to twice the angle  $\angle MOB$ ; and because  $\angle AMB$  is equal to  $\angle OAB$ , and also to  $\angle OBA$ , the sum of the angles  $\angle OAB$  and  $\angle OBA$  is equal to four times the angle  $\angle AOB$ : hence,  $\angle AOB$  is equal to one-fifth of two right angles, or to one-tenth of four right angles; and consequently, the arc  $AB$  is equal to one-tenth of the circumference: hence, the chord  $AB$  is equal to the side of a regular inscribed decagon; *which was to be proved.*



*Cor. 1.* If  $AB$  be applied ten times as a chord, the resulting polygon will be a regular inscribed decagon.

*Cor. 2.* If the vertices  $A, C, E, G$ , and  $I$ , of the alternate angles of the decagon be joined by straight lines, the resulting figure will be a regular inscribed pentagon.

*Scholium 1.* If the arcs subtended by the sides of any regular inscribed polygon be bisected, and chords of the semi-arcs be drawn, the resulting figure will be a regular inscribed polygon of double the number of sides.

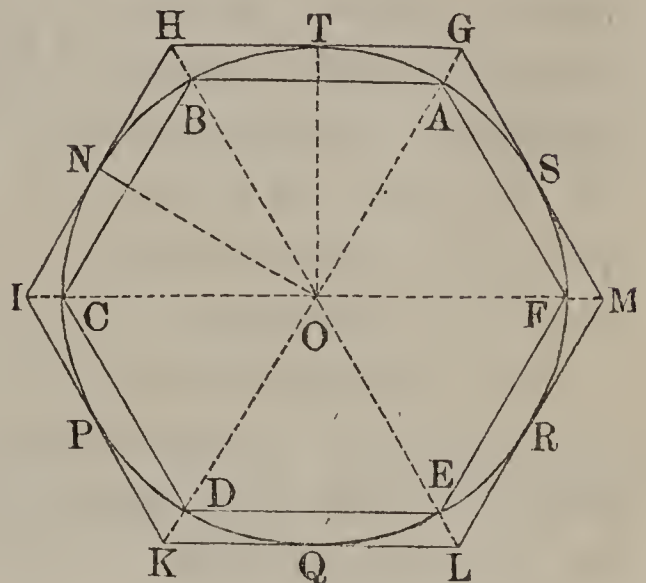
*Scholium 2.* The area of any regular inscribed polygon is less than that of a regular inscribed polygon of double the number of sides, because a part is less than the whole.

## PROPOSITION VII. PROBLEM.

*To circumscribe a polygon about a circle which shall be similar to a given regular inscribed polygon.*

Let  $TNQ$  be a circle,  $O$  its centre, and  $ABCDEF$ , a regular inscribed polygon.

At the middle points  $T$ ,  $N$ ,  $P$ , &c., of the arcs subtended by the sides of the inscribed polygon, draw tangents to the circle, and prolong them till they intersect; then will the resulting figure be the polygon required.



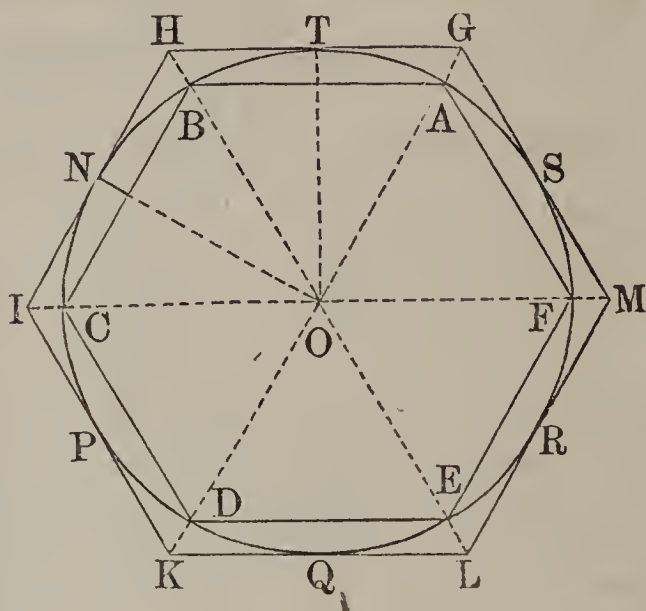
1°. The side  $HG$ , being parallel to  $BA$ , and

$HI$  to  $BC$ , the angle  $H$  is equal to the angle  $B$ . In like manner, it may be shown that any other angle of the circumscribed polygon is equal to the corresponding angle of the inscribed polygon: hence, the circumscribed polygon is *equiangular*.

2°. Draw the lines  $OG$ ,  $OT$ ,  $OH$ ,  $ON$ , and  $OI$ . Then, because the lines  $HT$  and  $HN$  are tangent to the circle,  $OH$  will bisect the angle  $NHT$ , and also the angle  $NOT$  (B. III., Prob. XIV., S.); consequently, it will pass through the middle point  $B$  of the arc  $NBT$ . In like manner, it may be shown that the line drawn from the centre to the vertex of any other angle of the circumscribed polygon, will pass through the corresponding vertex of the inscribed polygon.

The triangles  $OIG$  and  $OHI$  have the angles  $OIG$

and  $OHI$  equal, from what has just been shown; the angles  $GOH$  and  $HOI$  equal, because they are measured by the equal arcs  $AB$  and  $BC$ , and the side  $OH$  common; they are, therefore, equal in all their parts: hence,  $GH$  is equal to  $HI$ . In like manner, it may be shown that  $HI$  is equal to  $IK$ ,  $IK$  to  $KL$ ; and so on: hence, the circumscribed polygon is *equilateral*.



The circumscribed polygon being both equiangular and equilateral, is *regular*; and since it has the same number of sides as the inscribed polygon, it is similar to it.

*Cor. 1.* If lines be drawn from the centre of a regular circumscribed polygon to its vertices, and the consecutive points in which they intersect the circumference be joined by chords, the resulting figure will be a regular inscribed polygon similar to the given polygon.

*Cor. 2.* The sum of the lines  $HT$  and  $HN$  is equal to the sum of  $HT$  and  $TG$ , or to  $HG$ ; that is, to one of the sides of the circumscribed polygon.

*Cor. 3.* If at the vertices  $A$ ,  $B$ ,  $C$ , &c., of the inscribed polygon, tangents be drawn to the circle and prolonged till they meet the sides of the circumscribed polygon, the resulting figure will be a circumscribed polygon of double the number of sides.

*Cor. 4.* The area of any regular circumscribed polygon



is greater than that of a regular circumscribed polygon of double the number of sides, because the whole is greater than any of its parts.

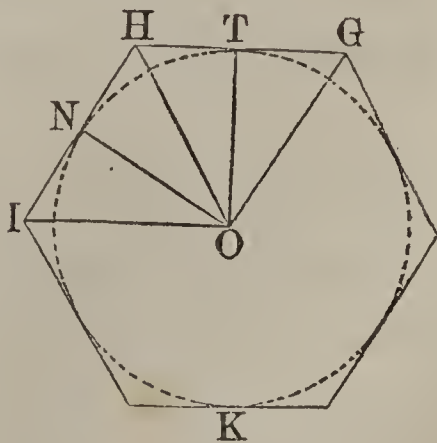
*Scholium.* By means of a circumscribed and inscribed square, we may construct, in succession, regular circumscribed and inscribed polygons of 8, 16, 32, &c., sides. By means of the regular hexagon, we may, in like manner, construct regular polygons of 12, 24, 48, &c., sides. By means of the decagon, we may construct regular polygons of 20, 40, 80, &c., sides.

PROPOSITION VIII. THEOREM.

*The area of a regular polygon is equal to half the product of its perimeter and apothem.*

Let  $GHIK$  be a regular polygon,  $O$  its centre, and  $OT$  its apothem, or the radius of the inscribed circle: then will the area of the polygon be equal to half the product of the perimeter and the apothem.

For, draw lines from the centre to the vertices of the polygon. These lines will divide the polygon into triangles whose bases will be the sides of the polygon, and whose altitudes will be equal to the apothem. Now, the area of any triangle, as  $OIG$ , is equal to half the product of the side  $IG$  and the apothem; hence, the area of the polygon is equal to half the product of the perimeter and the apothem; *which was to be proved.*

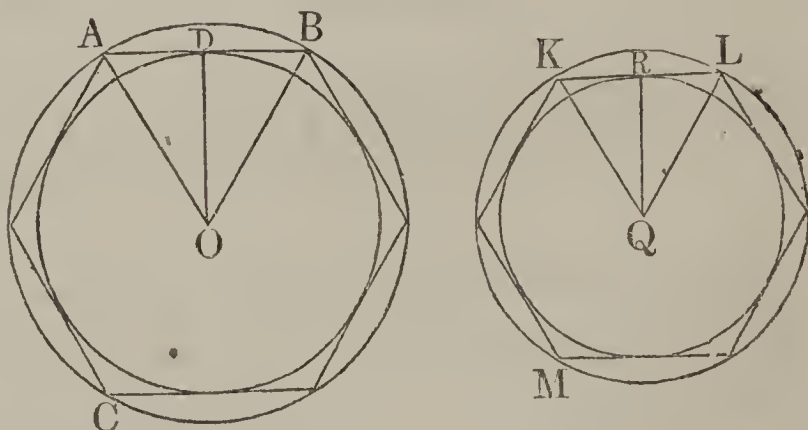


## PROPOSITION IX. THEOREM.

*The perimeters of similar regular polygons are to each other as the radii of their circumscribed or inscribed circles; and their areas are to each other as the squares of those radii.*

1°. Let  $ABC$  and  $KLM$  be similar regular polygons. Let  $OA$  and  $QK$  be the radii of their circumscribed,  $OD$  and  $QR$  be the radii of their inscribed circles: then will the perimeters of the polygons be to each other as  $OA$  is to  $QK$ , or as  $OD$  is to  $QR$ .

For, the lines  $OA$  and  $QK$  are homologous lines of the polygons to which they belong, as are also the lines  $OD$  and  $QR$ : hence, the perimeter of  $ABC$



is to the perimeter of  $KLM$ , as  $OA$  is to  $QK$ , or as  $OD$  is to  $QR$  (B. IV., P. XXVII., C. 1); *which was to be proved.*

2°. The areas of the polygons will be to each other as  $\overline{OA}^2$  is to  $\overline{QK}^2$ , or as  $\overline{OD}^2$  is to  $\overline{QR}^2$ .

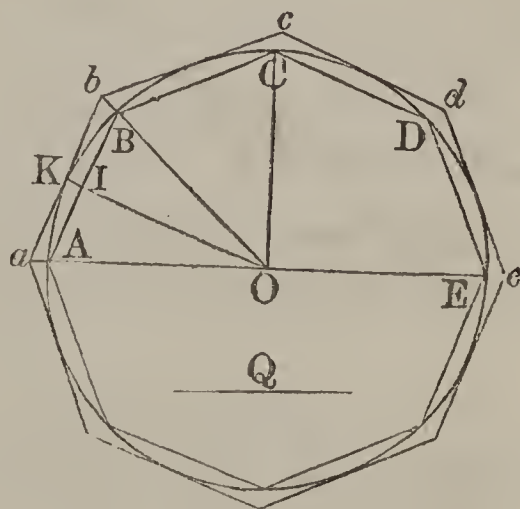
For,  $OA$  being homologous with  $QK$ , and  $OD$  with  $QR$ , we have, the area of  $ABC$  is to the area of  $KLM$ , as  $\overline{OA}^2$  is to  $\overline{QK}^2$ , or as  $\overline{OD}^2$  is to  $\overline{QR}^2$  (B. IV., P. XXVII., C. 1); *which was to be proved.*

## PROPOSITION X. THEOREM.

*Two regular polygons of the same number of sides can be constructed, the one circumscribed about a circle and the other inscribed in it, which shall differ from each other by less than any given surface.*

Let  $ABCE$  be a circle,  $O$  its centre, and  $Q$  the side of a square which is less than the given surface; then can two similar regular polygons be constructed, the one circumscribed about, and the other inscribed within the given circle, which shall differ from each other by less than the square of  $Q$ , and consequently, by less than the given surface.

Inscribe a square in the given circle (P. III.), and by means of it, inscribe, in succession, regular polygons of 8, 16, 32, &c., sides (P. VII., S.), until one is found whose side is less than  $Q$ ; let  $AB$  be the side of such a polygon.



Construct a similar circumscribed polygon  $abcde$ : then will these polygons differ from each other by less than the square of  $Q$ .

For, from  $a$  and  $b$ , draw the lines  $aO$  and  $bO$ ; they will pass through the points  $A$  and  $B$ . Draw also  $OK$  to the point of contact  $K$ ; it will bisect  $AB$  at  $I$  and be perpendicular to it. Prolong  $AO$  to  $E$ .

Let  $P$  denote the circumscribed, and  $p$  the inscribed polygon; then, because they are regular and similar, we shall have (P. IX.),



$$P : p :: \overline{OK}^2 \text{ or } \overline{OA}^2 : \overline{OI}^2;$$

hence, by division (B. II., P. VI.), we have,

$$P : P - p :: \overline{OA}^2 : \overline{OA}^2 - \overline{OI}^2;$$

or,

$$P : P - p :: \overline{OA}^2 : \overline{AI}^2.$$

Multiplying the terms of the second couplet by 4 (B. II., P. VII), we have,

$$P : P - p :: 4\overline{OA}^2 : 4\overline{AI}^2;$$

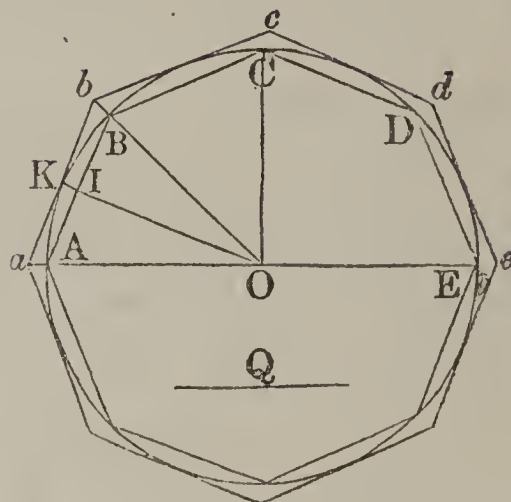
whence (B. IV., P. VIII., C.),

$$P : P - p :: \overline{AE}^2 : \overline{AB}^2.$$

But  $P$  is less than the square of  $AE$  (P. VII., C. 4); hence,  $P - p$  is less than the square of  $AB$ , and consequently, less than the square of  $Q$ , or than the given surface; *which was to be proved.*

*Cor. 1.* If the number of sides of the polygons be made greater than any assignable number; that is, *infinite*, the difference between their areas will be less than any assignable surface; that is, it will be *zero*\*.

*Cor. 2.* When the number of sides of the polygons is infinite, either polygon differs from the circle by less than any assignable quantity; for, the circumference of the circle lies between the perimeters of the polygons: hence, the circle differs from either polygon by less than they differ from each other.



\* Univ. Algebra, Arts. 72, 73. Bourdon, Art. 71:

*Scholium 1.* The circle may be regarded as the *limit* of the inscribed polygons; that is, it is a figure towards which a polygon may be made to approach as near as desirable, but beyond which it cannot be made to pass.

*Scholium 2.* The circle may be regarded as a regular polygon of an infinite number of sides, and because of the principle, that *whatever is true of a whole class, is true of every individual of that class*, we may affirm that *whatever is true of regular polygons, is also true of circles*.

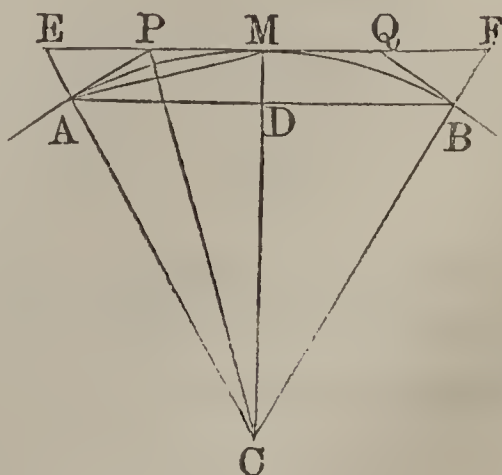
*Scholium 3.* When the circle is regarded as a regular polygon, the circumference is to be regarded as its *perimeter*, and the radius as its *apothem*.

PROPOSITION XI. PROBLEM.

*The area of a regular inscribed polygon, and that of a similar circumscribed polygon being given, to find the areas of the regular inscribed and circumscribed polygons having double the number of sides.*

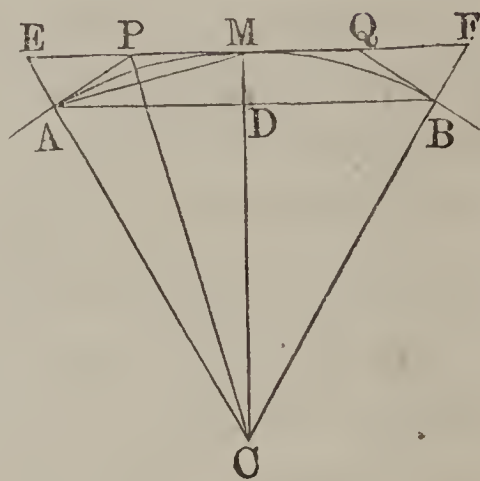
Let  $AB$  be the side of the given inscribed, and  $EF$  that of the given circumscribed polygon. Let  $C$  be their common centre,  $AMB$  a portion of the circumference of the circle, and  $M$  the middle point of the arc  $AMB$ .

Draw the chord  $AM$ , and at  $A$  and  $B$  draw the tangents  $AP$  and  $BQ$ ; then will  $AM$  be the side of the inscribed polygon, and  $PQ$  the side of the circumscribed polygon of double the number of sides (P. VII.). Draw  $CE$ ,  $CP$ ,  $CM$ , and  $CF$ .



Denote the area of the given inscribed polygon by  $p$ , the area of the given circumscribed polygon by  $P$ , and the areas of the inscribed and circumscribed polygons having double the number of sides, respectively by  $p'$  and  $P'$ .

1°. The triangles  $CAD$ ,  $CAM$ , and  $CEM$ , are like parts of the polygons to which they belong: hence, they are proportional to the polygons themselves. But  $CAM$  is a mean proportional between  $CAD$  and  $CEM$  (B. IV., P. XXIV., C. 2); consequently  $p'$  is a mean proportional between  $p$  and  $P$ : hence,



$$p' = \sqrt{p \times P} \dots \dots \dots (1.)$$

2°. Because the triangles  $CPM$  and  $CPE$  have the common altitude  $CM$ , they are to each other as their bases: hence,

$$CPM : CPE :: PM : PE;$$

and because  $CP$  bisects the angle  $ACM$ , we have (B. IV., P. XVII.),

$$PM : PE :: CM : CE :: CD : CA;$$

hence (B. II., P. II.),

$$CPM : CPE :: CD : CA \text{ or } CM.$$

But, the triangles  $CAD$  and  $CAM$  have the common altitude  $AD$ ; they are, therefore, to each other as their bases: hence,

$$CAD : CAM :: CD : CM;$$



or, because  $CAD$  and  $CAM$  are to each other as the polygons to which they belong,

$$p : p' :: CD : CM ;$$

hence (B. II., P. IV.), we have,

$$CPM : CPE :: p : p',$$

and, by composition,

$$CPM : CPM + CPE \text{ or } CME :: p : p + p' ;$$

hence (B. II., P. VII.),

$$2CPM \text{ or } CMPA : CME :: 2p : p + p'.$$

But,  $CMPA$  and  $CME$  are like parts of  $P'$  and  $P$ , hence,

$$P' : P :: 2p : p + p' ;$$

or,

$$P' = \frac{2p \times P}{p + p'} . . . . . (2.)$$

*Scholium.* By means of Equation (1), we can find  $p'$ , and then, by means of Equation (2), we can find  $P'$ .

#### PROPOSITION XII. PROBLEM.

*To find the approximate area of a circle whose radius is 1.*

The area of an inscribed square is equal to twice the square of the radius, or 2 (P. III., S.), and the area of a circumscribed square is 4. Making  $p$  equal to 2, and  $P$  equal to 4, we have, from Equations (1) and (2) of Proposition XI.,

$$p' = \sqrt{8} = 2.8284271 . . . \text{ inscribed octagon ;}$$

$$P' = \frac{16}{2 + \sqrt{8}} = 3.3137085 . . . \text{ circumscribed octagon.}$$

Making  $p$  equal to 2.8284271, and  $P$  equal to 3.3137085, we have, from the same equations,

$$p' = 3.0614674 \quad . \quad . \quad . \quad \text{inscribed polygon of 16 sides.}$$

$$P' = 3.1825979 \quad . \quad . \quad . \quad \text{circumscribed polygon of 16 sides.}$$

By a continued application of these equations, we find the areas indicated in the following

TABLE.

NUMBER OF SIDES.	INSCRIBED POLYGONS. -	CIRCUMSCRIBED POLYGONS.
4 . .	2.0000000 . .	4.0000000
8 . .	2.8284271 . .	3.3137085
16 . .	3.0614674 . .	3.1825979
32 . .	3.1214451 . .	3.1517249
64 . .	3.1365485 . .	3.1441184
128 . .	3.1403311 . .	3.1422236
256 . .	3.1412772 . .	3.1417504
512 . .	3.1415138 . .	3.1416321
1024 . .	3.1415729 . .	3.1416625
2048 . .	3.1415877 . .	3.1415951
4096 . .	3.1415914 . .	3.1415933
8192 . .	3.1415923 . .	3.1415928
16384 . .	3.1415925 . .	3.1415927

Now, the areas of the last two polygons differ from each other by less than the millionth part of a unit, but the area of the circle differs from either by less than they differ from each other; hence, the value of the area of either will differ from that of the circle by less than a millionth part of a unit. Taking the figures as far as they agree, and denoting the number of units in the required area by  $\pi$ , we have, approximately,

$$\pi = 3.141592 ;$$

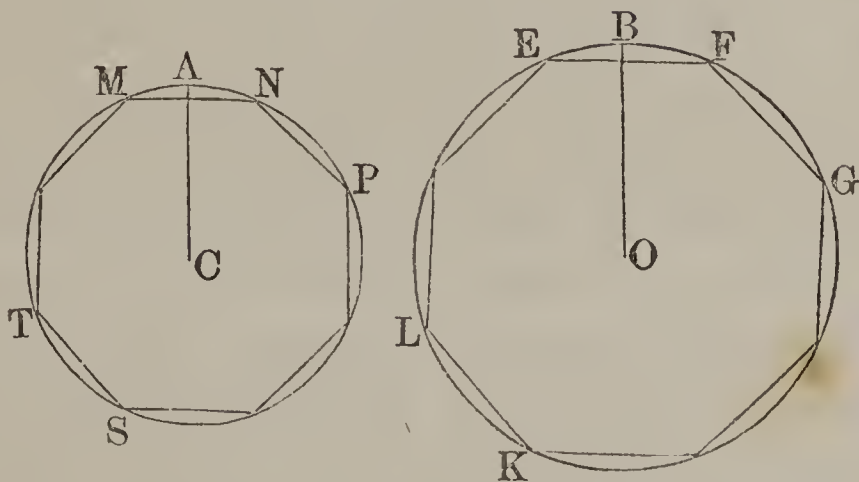
that is, the area of a circle whose radius is 1, is 3.141592.

*Scholium.* For practical computation, the value of  $\pi$  is taken equal to 3.1416.

## PROPOSITION XIII. THEOREM.

*The circumferences of circles are to each other as their radii, and the areas are to each other as the squares of their radii.*

Let  $C$  and  $O$  be the centres of two circles whose radii are  $CA$  and  $OB$ : then will the circumferences be to each other as their radii, and the areas will be to each other as the squares of their radii.



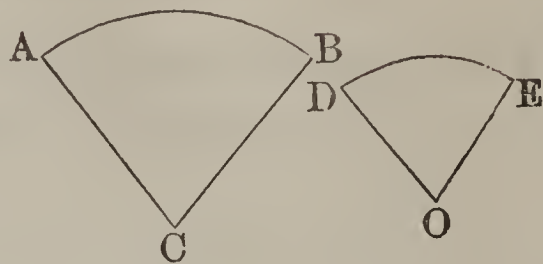
For, let similar regular polygons  $MNPST$  and  $EFGKL$  be inscribed in the circles: then will the perimeters of these polygons be to each other as their apothems, and the areas will be to each other as the squares of their apothems, whatever may be the number of their sides (P. IX.).

If the number of sides be made infinite (P. VII., S.), the polygons will coincide with the circles, the perimeters with the circumferences, and the apothems with the radii: hence, the circumferences of the circles are to each other as their radii, and the areas are to each other as the squares of the radii; *which was to be proved.*

*Cor. 1.* Diameters of circles are proportional to their radii: hence, *the circumferences of circles are proportional to their diameters, and the areas are proportional to the squares of the diameters.*



*Cor. 2.* Similar arcs, as  $AB$  and  $DE$ , are like parts of the circumferences to which they belong, and similar sectors, as  $ACB$  and  $DOE$ , are like parts of the circles to which they belong : hence, *similar arcs are to each other as their radii, and similar sectors are to each other as the squares of their radii.*



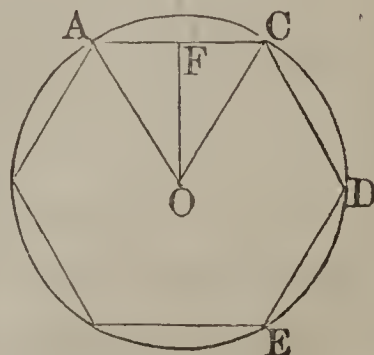
*Scholium.* The term *infinite*, employed above, is to be understood in its limited *technical sense*. When it is proposed to make the number of sides of the polygons infinite, by the method indicated in the Scholium of Proposition VII., it is simply meant to make that number so great that the difference between the areas of the circle and polygon shall be less than any appreciable quantity. We have seen (P. XII.), that when the number of sides 16384, the areas differ by less than the millionth part of a unit. By increasing the number of sides, a still closer approximation may be had.

#### PROPOSITION XIV. THEOREM.

*The area of a circle is equal to half the product of its circumference and radius.*

Let  $O$  be the centre of a circle,  $OC$  its radius, and  $ACDE$  its circumference : then will the area of the circle be equal to half the product of the circumference and radius.

For, inscribe in it a regular polygon  $ACDE$ . Then will the area of this polygon be equal to half the pro-



duct of its perimeter and apothem, whatever may be the number of its sides (P. VIII.).

If the number of sides be made infinite, the polygon will coincide with the circle, the perimeter with the circumference, and the apothem with the radius : hence, the area of the circle is equal to half the product of its circumference and radius ; *which was to be proved.*

*Cor. 1.* The area of a sector is equal to half the product of its arc and radius.

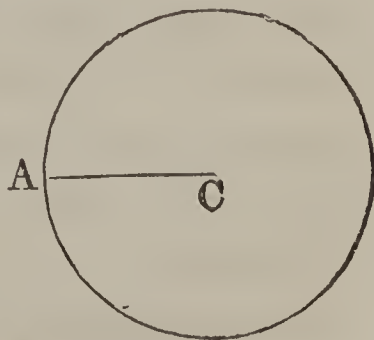
*Cor. 2.* The area of a sector is to the area of the circle, as the arc of the sector to the circumference.

PROPOSITION XV. PROBLEM.

*To find an expression for the area of any circle in terms of its radius.*

Let  $C$  be the centre of a circle, and  $CA$  its radius. Denote its area by *area*  $CA$ , its radius by  $R$ , and the area of a circle whose radius is 1, by  $\pi$  (P. XII., S.).

Then, because the areas of circles are to each other as the squares of their radii (P. XIII.), we have,



$$\text{area } CA : \pi :: R^2 : 1 ;$$

whence,

$$\text{area } CA = \pi R^2.$$

That is, *the area of any circle is 3.1416 times the square of the radius.*

PROPOSITION XVI. PROBLEM.

*To find an expression for the circumference of a circle, in terms of its radius, or diameter.*

Let  $C$  be the centre of a circle, and  $CA$  its radius.

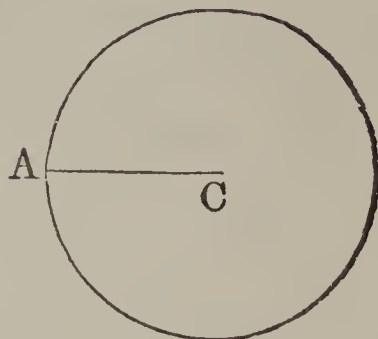
Denote its circumference by *circ. CA*, its radius by  $R$ , and its diameter by  $D$ . From the last Proposition, we have,

$$\text{area } CA = \pi R^2 ;$$

and, from Proposition XIV., we have,

$$\text{area } CA = \frac{1}{2} \text{circ. } CA \times R ;$$

hence,  $\frac{1}{2} \text{circ. } CA \times R = \pi R^2 ;$



whence, by reduction,

$$\text{circ. } CA = 2\pi R, \text{ or, } \text{circ. } CA = \pi D.$$

That is, *the circumference of any circle is equal to 3.1416 times its diameter.*

*Scholium 1.* The abstract number  $\pi$ , equal to 3.1416, denotes the number of times that the diameter of a circle is contained in the circumference, and also the number of times that the square constructed on the radius is contained in the area of the circle (P. XV.). Now, it has been proved by the methods of Higher Mathematics, that the value of  $\pi$  is incommensurable with 1 ; hence, it is impossible to express, by means of numbers, the exact length of a circumference in terms of the radius, or the exact area in terms of the square described on the radius. We may also infer that it is impossible to *square the circle* ; that is, to construct a square whose area shall be exactly equal to that of the circle.

*Scholium 2.* Besides the approximate value of  $\pi$ , 3.1416, usually employed, the fractions  $\frac{22}{7}$  and  $\frac{355}{113}$  are also used, when great accuracy is not required.



## BOOK VI.

### PLANES AND POLYEDRAL ANGLES.

#### DEFINITIONS.

1. A straight line is PERPENDICULAR TO A PLANE, when it is perpendicular to every line of the plane which passes through its FOOT; that is, through the *point* in which it meets the plane.

In this case, the plane is also perpendicular to the line.

2. A straight line is PARALLEL TO A PLANE, when it cannot meet the plane, how far soever both may be produced.

In this case, the plane is also parallel to the line.

3. TWO PLANES ARE PARALLEL, when they cannot meet, how far soever both may be produced.

4. A DIEDRAL ANGLE is the amount of divergence of two planes.

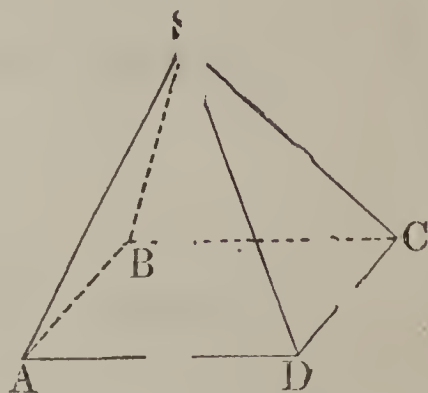
The line in which the planes meet, is called the *edge of the angle*, and the planes themselves are called *faces of the angle*.

The measure of a diedral angle is the same as that of a plane angle formed by two lines, one drawn in each face, and both perpendicular to the edge at the same point. A diedral angle may be *acute*, *obtuse*, or a *right angle*. In the latter case, the faces are *perpendicular* to each other.

5. A POLYEDRAL ANGLE is the amount of divergence of several planes meeting at a common point.

This point is called the *vertex of the angle*; the lines in which the planes meet are called *edges of the angle*, and the portions of the planes lying between the edges are called *faces of the angle*. Thus,  $S$  is the vertex of the polyedral angle, whose edges are  $SA$ ,  $SB$ ,  $SC$ ,  $SD$ , and whose faces are  $ASB$ ,  $BSC$ ,  $CSD$ ,  $DSA$ .

A polyedral angle which has but three faces, is called a *triedral angle*.



#### POSTULATE.

A line may be drawn perpendicular to a plane from any point of the plane, or from any point without the plane.

#### PROPOSITION I. THEOREM.

*If a straight line has two of its points in a plane, it will lie wholly in that plane.*

For, by definition, a plane is a surface such, that if any two of its points be joined by a straight line, that line will lie wholly in the surface (B. I., D. 8).

*Cor.* Through any point of a plane, an infinite number of straight lines may be drawn which will lie in the plane. For, if a line be drawn from the given point to any other point of the plane, that line will lie wholly in the plane.

*Scholium.* If any two points of a plane be joined by a straight line, the plane may be turned about that line as an

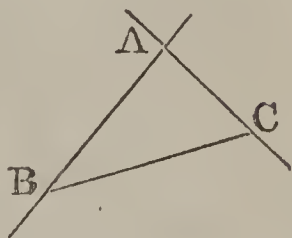
axis, so as to take an infinite number of positions. Hence, we infer that an infinite number of planes may be passed through a given line.

PROPOSITION II. THEOREM.

*Through three points, not in the same straight line, one plane can be passed, and only one.*

Let  $A$ ,  $B$ , and  $C$  be the three points: then can one plane be passed through them, and only one.

Join two of the points, as  $A$  and  $B$ , by the line  $AB$ . Through  $AB$  let a plane be passed, and let this plane be turned around  $AB$  until it contains the point  $C$ ; in this position it will pass through the three points  $A$ ,  $B$ , and  $C$ . If now, the plane be turned about  $AB$ , in either direction, it will no longer contain the point  $C$ : hence, one plane can always be passed through three points, and only one; *which was to be proved.*



*Cor. 1.* Three points, not in a straight line, determine the position of a plane, because only one plane can be passed through them.

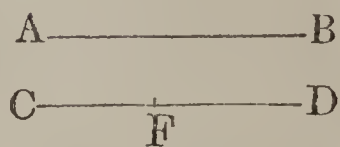
*Cor. 2.* A straight line and a point without that line, determine the position of a plane, because only one plane can be passed through them.

*Cor. 3.* Two straight lines which intersect, determine the position of a plane. For, let  $AB$  and  $AC$  intersect at  $A$ : then will either line, as  $AB$ , and one point of the other, as  $C$ , determine the position of a plane.

*Cor. 4.* Two parallel lines determine the position of a



plane. For, let  $AB$  and  $CD$  be parallel. By definition (B. I., D. 16) two parallel lines always lie in the same plane. But either line, as  $AB$ , and any point of the other, as  $F$ , determine the position of a plane : hence, two parallels determine the position of a plane.

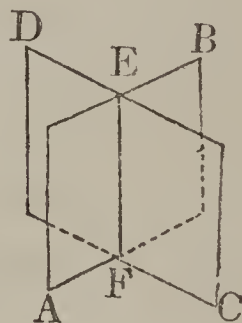


PROPOSITION III. THEOREM.

*The intersection of two planes is a straight line.*

Let  $AB$  and  $CD$  be two planes : then will their intersection be a straight line.

For, let  $E$  and  $F$  be any two points common to the planes ; draw the straight line  $EF$ . This line having two points in the plane  $AB$ , will lie wholly in that plane ; and having two points in the plane  $CD$ , will lie wholly in that plane : hence, every point of  $EF$  is common to both planes. Furthermore, the planes can have no common point lying without  $EF$ , otherwise there would be two planes passing through a straight line and a point lying without it, which is impossible (P. II., C. 2) ; hence, the intersection of the two planes is a straight line ; *which was to be proved.*



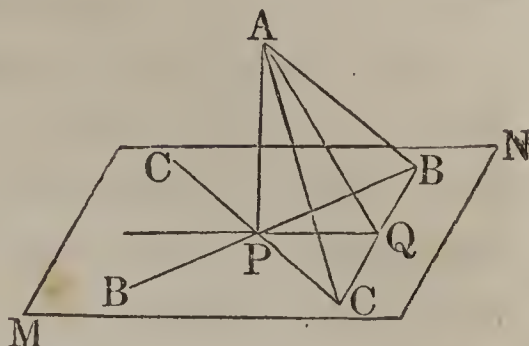
PROPOSITION IV. THEOREM.

*If a straight line is perpendicular to two straight lines at their point of intersection, it is perpendicular to the plane of those lines.*

Let  $MN$  be the plane of the two lines  $BB$ ,  $CC$ , and let  $AP$  be perpendicular to these lines at  $P$  : then will

$AP$  be perpendicular to every line of the plane which passes through  $P$ , and consequently, to the plane itself.

For, through  $P$ , draw in the plane  $MN$ , any line  $PQ$ ; through any point of this line, as  $Q$ , draw the line  $BC$ , so that  $BQ$  shall be equal to  $QC$  (B. IV., Prob. V.); draw  $AB$ ,  $AQ$ , and  $AC$ .



The base  $BC$ , of the triangle  $BPC$ , being bisected at  $Q$ , we have (B. IV., P. XIV.),

$$\overline{PC}^2 + \overline{PB}^2 = 2\overline{PQ}^2 + 2\overline{QC}^2.$$

In like manner, we have, from the triangle  $ABC$ ,

$$\overline{AC}^2 + \overline{AB}^2 = 2\overline{AQ}^2 + 2\overline{QC}^2.$$

Subtracting the first of these equations from the second, member from member, we have,

$$\overline{AC}^2 - \overline{PC}^2 + \overline{AB}^2 - \overline{PB}^2 = 2\overline{AQ}^2 - 2\overline{PQ}^2.$$

But, from Proposition XI., C. 1, Book IV., we have,

$$\overline{AC}^2 - \overline{PC}^2 = \overline{AP}^2, \quad \text{and} \quad \overline{AB}^2 - \overline{PB}^2 = \overline{AP}^2;$$

hence, by substitution,

$$2\overline{AP}^2 = 2\overline{AQ}^2 - 2\overline{PQ}^2;$$

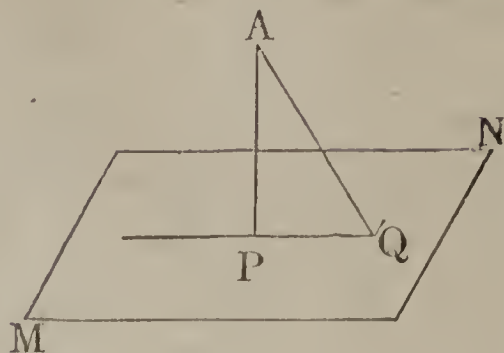
whence,

$$\overline{AP}^2 = \overline{AQ}^2 - \overline{PQ}^2; \quad \text{or,} \quad \overline{AP}^2 + \overline{PQ}^2 = \overline{AQ}^2.$$

The triangle  $APQ$  is, therefore, right-angled at  $P$  (B. IV., P. XIII., S.), and consequently,  $AP$  is perpendicular to  $PQ$ : hence,  $AP$  is perpendicular to every line of the plane  $MN$  passing through  $P$ , and consequently, to the plane itself; *which was to be proved.*

*Cor. 1.* Only one perpendicular can be drawn to a plane from a point without the plane.

For, suppose two perpendiculars, as  $AP$  and  $AQ$ , could be drawn from the point  $A$  to the plane  $MN$ . Draw  $PQ$ ; then the triangle  $APQ$  would have two right angles,  $APQ$  and  $AQP$ ; which is impossible (B. I., P. XXV., C. 3).



*Cor. 2.* Only one perpendicular can be drawn to a plane from a point of that plane. For, suppose that two perpendiculars could be drawn to the plane  $MN$ , from the point  $P$ . Pass a plane through the perpendiculars, and let  $PQ$  be its intersection with  $MN$ ; then we should have two perpendiculars drawn to the same straight line from a point of that line; which is impossible (B. I., P. XIV., C.).

#### PROPOSITION V. THEOREM.

*If from a point without a plane, a perpendicular be drawn to the plane, and oblique lines be drawn to different points of the plane :*

- 1°. *The perpendicular will be shorter than any oblique line :*
- 2°. *Oblique lines which meet the plane at equal distances from the foot of the perpendicular, will be equal :*
- 3°. *Of two oblique lines which meet the plane at unequal distances from the foot of the perpendicular, the one which meets it at the greater distance will be the longer.*

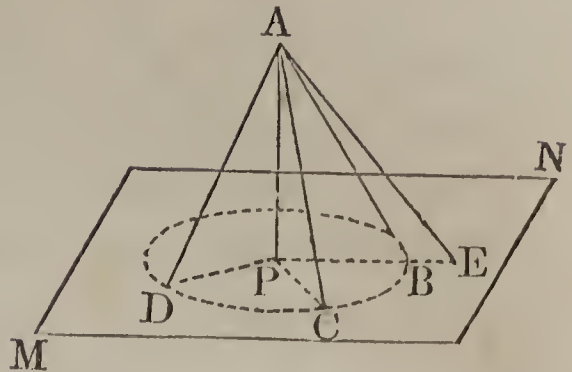
Let  $A$  be a point without the plane  $MN$ ; let  $AP$  be perpendicular to the plane; let  $AC$ ,  $AD$ , be any two oblique lines meeting the plane at equal distances from the foot of the perpendicular; and let  $AC$  and  $AE$  be any



two oblique lines meeting the plane at unequal distances from the foot of the perpendicular :

1°.  $AP$  will be shorter than any oblique line  $AC$ .

For, draw  $PC$ ; then will  $AP$  be less than  $AC$  (B. I., P. XV.) ; *which was to be proved.*



2°.  $AC$  and  $AD$  will be equal.

For, draw  $PD$ ; then the right-angled triangles  $APC$ ,  $APD$ , will have the side  $AP$  common, and the sides  $PC$ ,  $PD$ , equal : hence, the triangles are equal in all their parts, and consequently,  $AC$  and  $AD$  will be equal ; *which was to be proved.*

3°.  $AE$  will be greater than  $AC$ .

For, draw  $PE$ , and take  $PB$  equal to  $PC$ ; draw  $AB$  : then will  $AE$  be greater than  $AB$  (B. I., P. XV.) ; but  $AB$  and  $AC$  are equal : hence,  $AE$  is greater than  $AC$  ; *which was to be proved.*

*Cor.* The equal oblique lines  $AB$ ,  $AC$ ,  $AD$ , meet the plane  $MN$  in the circumference of a circle, whose centre is  $P$ , and whose radius is  $PB$  : hence, to draw a perpendicular to a given plane  $MN$ , from a point  $A$ , without that plane, find three points  $B$ ,  $C$ ,  $D$ , of the plane equally distant from  $A$ , and then find the centre  $P$ , of the circle whose circumference passes through these points : then will  $AP$  be the perpendicular required.

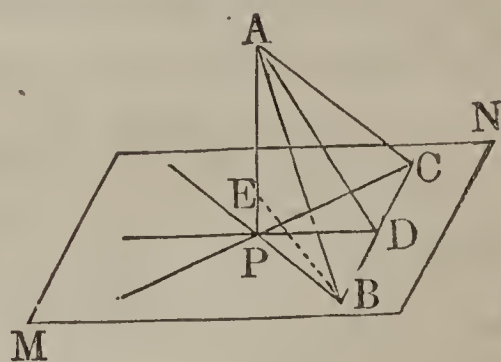
*Scholium.* The angle  $ABP$  is called *the inclination of the oblique line  $AB$  to the plane  $MN$ .* The equal oblique lines  $AB$ ,  $AC$ ,  $AD$ , are all equally inclined to the plane  $MN$ . The inclination of  $AE$  is less than the inclination of any shorter line  $AB$ .

## PROPOSITION VI. THEOREM.

*If from the foot of a perpendicular to a plane, a line be drawn at right angles to any line of that plane, and the point of intersection be joined with any point of the perpendicular, the last line will be perpendicular to the line of the plane.*

Let  $AP$  be perpendicular to the plane  $MN$ ,  $P$  its foot,  $BC$  the given line, and  $A$  any point of the perpendicular; draw  $PD$  at right angles to  $BC$ , and join the point  $D$  with  $A$ : then will  $AD$  be perpendicular to  $BC$ .

For, lay off  $DB$  equal to  $DC$ , and draw  $PB$ ,  $PC$ ,  $AB$ , and  $AC$ . Because  $PD$  is perpendicular to  $BC$ , and  $DB$  equal to  $DC$ , we have,  $PB$  equal to  $PC$  (B. I., P. XV.); and because  $AP$  is perpendicular to the plane  $MN$ , and  $PB$



equal to  $PC$ , we have  $AB$  equal to  $AC$  (P. V.). The line  $AD$  has, therefore, two of its points  $A$  and  $D$ , each equally distant from  $B$  and  $C$ : hence, it is perpendicular to  $BC$  (B. I., P. XVI., S.); *which was to be proved.*

*Cor. 1.* The line  $BC$  is perpendicular to the plane of the triangle  $APD$ ; because it is perpendicular to  $AD$  and  $PD$ , at  $D$  (P. IV.).

*Cor. 2.* The shortest distance between  $AP$  and  $BC$  is measured on  $PD$ , perpendicular to both. For, draw  $BE$  between any other points of the lines: then will  $BE$  be greater than  $PB$ , and  $PB$  will be greater than  $PD$ : hence,  $PD$  is less than  $BE$ .

*Scholium.* The lines  $AP$  and  $BC$ , though not in the same plane, are considered perpendicular to each other. In general, any two straight lines not in the same plane, are considered as making an angle with each other, which angle is equal to that formed by drawing through a given point, two lines respectively parallel to the given lines.

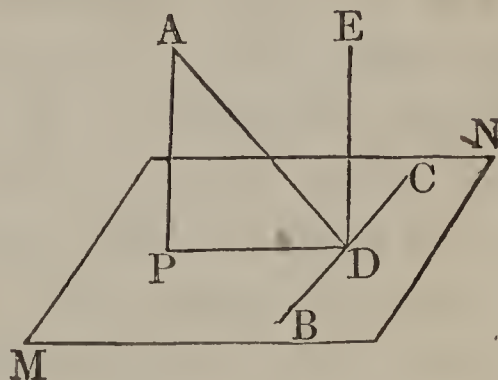
## PROPOSITION VII. THEOREM.

*If one of two parallels is perpendicular to a plane, the other one is also perpendicular to the same plane.*

Let  $AP$  and  $ED$  be two parallels, and let  $AP$  be perpendicular to the plane  $MN$ : then will  $ED$  be also perpendicular to the plane  $MN$ .

For, pass a plane through the parallels; its intersection with  $MN$  will be  $PD$ ; draw  $AD$ , and in the plane  $MN$  draw  $BC$  perpendicular to  $PD$  at  $D$ . Now,  $BD$  is perpendicular to the plane  $APDE$  (P. VI., C.);

the angle  $BDE$  is consequently a right angle; but the angle  $EDP$  is a right angle, because  $ED$  is parallel to  $AP$  (B. I., P. XX., C. 1): hence,  $ED$  is perpendicular to  $BD$  and  $PD$ , at their point of intersection, and consequently, to their plane  $MN$  (P. IV.); which was to be proved.



*Cor. 1.* If the lines  $AP$  and  $ED$  are perpendicular to the plane  $MN$ , they are parallel to each other. For, if not, draw through  $D$  a line parallel to  $PA$ ; it will be perpendicular to the plane  $MN$ , from what has just been proved; we shall, therefore, have two perpendiculars to the plane  $MN$ , at the same point; which is impossible (P. IV., C. 2).



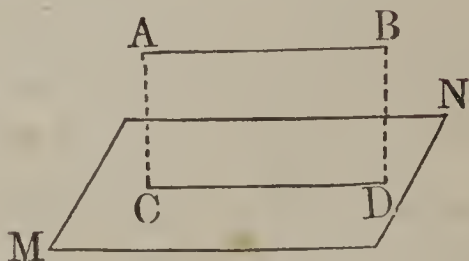
*Cor. 2.* If two lines,  $A$  and  $B$ , are parallel to a third line  $C$ , they are parallel to each other. For, pass a plane perpendicular to  $C$ ; it will be perpendicular to both  $A$  and  $B$ : hence,  $A$  and  $B$  are parallel.

PROPOSITION VIII. THEOREM.

*If a line is parallel to a line of a plane, it is parallel to that plane.*

Let the line  $AB$  be parallel to the line  $CD$  of the plane  $MN$ ; then will  $AB$  be parallel to the plane  $MN$ .

For, through  $AB$  and  $CD$  pass a plane (P. II., C. 4);  $CD$  will be its intersection with the plane  $MN$ . Now, since  $AB$  lies in this plane, if it can meet the plane  $MN$ , it will be at some point of  $CD$ ; but this is impossible, because  $AB$  and  $CD$  are parallel: hence,  $AB$  cannot meet the plane  $MN$ , and consequently, it is parallel to it; *which was to be proved.*

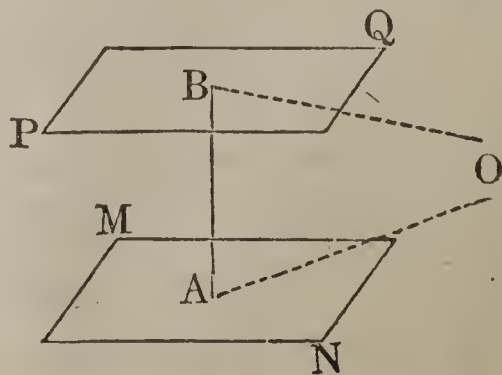


PROPOSITION IX. THEOREM.

*If two planes are perpendicular to the same straight line, they are parallel to each other.*

Let the planes  $MN$  and  $PQ$  be perpendicular to the line  $AB$ , at the points  $A$  and  $B$ : then will they be parallel to each other.

For, if they are not parallel,



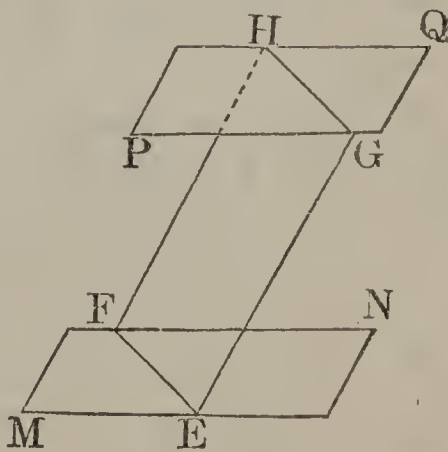
they will meet; and let  $O$  be a point common to both. From  $O$  draw the lines  $OA$  and  $OB$ : then, since  $OA$  lies in the plane  $MN$ , it will be perpendicular to  $BA$  at  $A$  (D. 1). For a like reason,  $OB$  will be perpendicular to  $AB$  at  $B$ : hence, the triangle  $OAB$  will have two right angles, which is impossible; consequently, the planes cannot meet, and are therefore parallel; *which was to be proved.*

## PROPOSITION X. THEOREM.

*If a plane intersect two parallel planes, the lines of intersection will be parallel.*

Let the plane  $EH$  intersect the parallel planes  $MN$  and  $PQ$ , in the lines  $EF$  and  $GH$ : then will  $EF$  and  $GH$  be parallel.

For, if they are not parallel, they will meet if sufficiently prolonged, because they lie in the same plane; but if the lines meet, the planes  $MN$  and  $PQ$ , in which they lie, will also meet; but this is impossible, because these planes are parallel: hence, the lines  $EF$  and  $GH$  cannot meet; they are, therefore, parallel; *which was to be proved.*

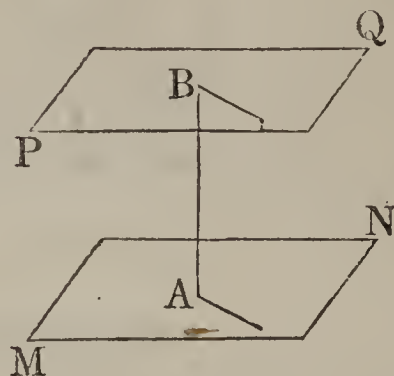


## PROPOSITION XI. THEOREM.

*If a straight line is perpendicular to one of two parallel planes, it is also perpendicular to the other.*

Let  $MN$  and  $PQ$  be two parallel planes, and let the line  $AB$  be perpendicular to  $PQ$ : then will it also be perpendicular to  $MN$ .

For, through  $AB$  pass any plane; its intersections with  $MN$  and  $PQ$  will be parallel (P. X.); but, its intersection with  $PQ$  is perpendicular to  $AB$  at  $B$  (D. 1); hence, its intersection with  $MN$  is also perpendicular to  $AB$  at  $A$  (B. I., P. XX., C. 1): hence,  $AB$  is perpendicular to every line of the plane  $MN$  through  $A$ , and is, therefore, perpendicular to that plane; *which was to be proved.*

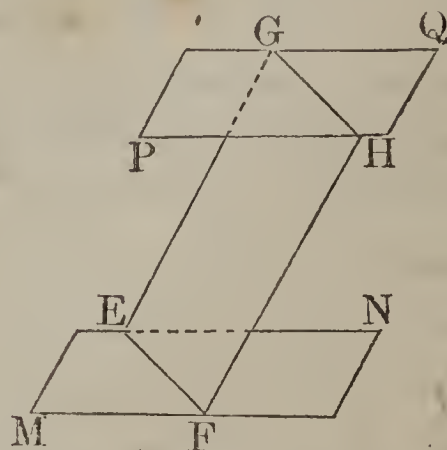


PROPOSITION XII. THEOREM.

*Parallel lines included between parallel planes, are equal.*

Let  $EG$  and  $FH$  be any two parallel lines included between the parallel planes  $MN$  and  $PQ$ : then will they be equal.

Through the parallels conceive a plane to be passed; it will intersect the plane  $MN$  in the line  $EF$ , and  $PQ$  in the line  $GH$ ; and these lines will be parallel (Prop. X.). The figure  $EFHG$  is, therefore, a parallelogram: hence,  $GE$  and  $HF$  are equal (B. I., P. XXVIII.); *which was to be proved.*



*Cor. 1.* The distance between two parallel planes is measured on a perpendicular to both; but any two perpendiculars between the planes are equal: hence, parallel planes are everywhere equally distant.

*Cor. 2.* If a line  $GH$  is parallel to any plane  $MN$ , then can a plane be passed through  $GH$  parallel to  $MN$ : hence, if a line is parallel to a plane, all of its points are equally distant from that plane.





ively to  $AB$  (P. XII.), and consequently,  $GD$  will be equal to  $CD$ , and  $HF$  to  $EF$ ; which is impossible: hence, the planes  $MN$  and  $PQ$  must be parallel; *which was to be proved.*

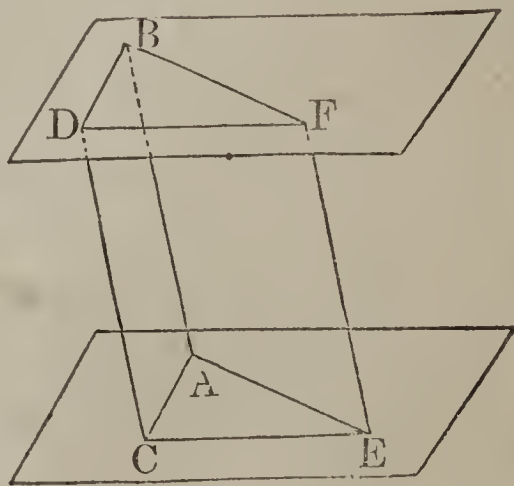
*Cor.* If two parallel planes  $MN$  and  $PQ$ , are met by two other planes  $AD$  and  $AF$ , the angles  $CAE$  and  $DBF$ , formed by their intersections, will be equal.

PROPOSITION XIV. THEOREM.

*If three straight lines, not situated in the same plane, are equal and parallel, the triangles formed by joining the extremities of these lines will be equal, and their planes parallel.*

Let  $AB$ ,  $CD$ , and  $EF$  be equal parallel lines not in the same plane: then will the triangles  $ACE$  and  $BDF$  be equal, and their planes parallel.

For,  $AB$  being equal and parallel to  $EF$ , the figure  $ABFE$  is a parallelogram, and consequently,  $AE$  is equal and parallel to  $BF$ . For a like reason,  $AC$  is equal and parallel to  $BD$ : hence, the included angles  $CAE$  and  $DBF$  are equal and their planes parallel (P. XIII.). Now, the triangles  $CAE$  and  $DBF$  have two sides and their



included angles equal, each to each: hence, they are equal in all their parts. The triangles are, therefore, equal and their planes parallel; *which was to be proved.*

## PROPOSITION XV. THEOREM.

*If two straight lines are cut by three parallel planes, they will be divided proportionally.*

Let the lines  $AB$  and  $CD$  be cut by the parallel planes  $MN$ ,  $PQ$ , and  $RS$ , in the points  $A$ ,  $E$ ,  $B$ , and  $C$ ,  $F$ ,  $D$ ; then

$$AE : EB :: CF : FD.$$

For, draw the line  $AD$ , and suppose it to pierce the plane  $PQ$  in  $G$ ; draw  $AC$ ,  $BD$ ,  $EG$ , and  $GF$ .

The plane  $ABD$  intersects the parallel planes  $RS$  and  $PQ$  in the lines  $BD$  and  $EG$ ; consequently, these lines are parallel (P. X.) : hence (B. IV., P. XV.),

$$AE : EB :: AG : GD.$$

The plane  $ACD$  intersects the parallel planes  $MN$  and  $PQ$ , in the parallel lines  $AC$  and  $GF$ : hence,

$$AG : GD :: CF : FD.$$

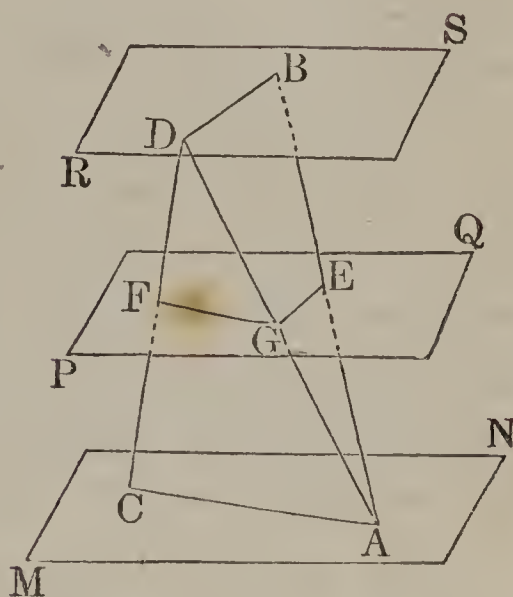
Combining these proportions (B. II., P. IV.), we have,

$$AE : EB :: CF : FD ;$$

*which was to be proved.*

*Cor. 1.* If two lines are cut by any number of parallel planes they will be divided proportionally.

*Cor. 2.* If any number of lines are cut by three parallel planes, they will be divided proportionally.



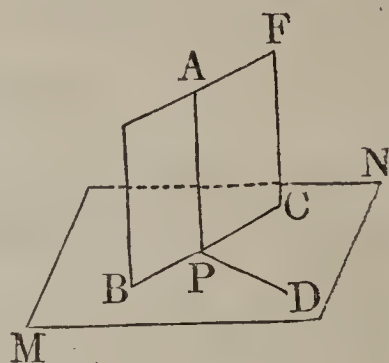


## PROPOSITION XVI. THEOREM.

*If a line is perpendicular to a plane, every plane passed through the line will also be perpendicular to that plane.*

Let  $AP$  be perpendicular to the plane  $MN$ , and let  $BF$  be a plane passed through  $AP$ : then will  $BF$  be perpendicular to  $MN$ .

In the plane  $MN$ , draw  $PD$  perpendicular to  $BC$ , the intersection of  $BF$  and  $MN$ . Since  $AP$  is perpendicular to  $MN$ , it is perpendicular to  $BC$  and  $DP$  (D. 1); and since  $AP$  and  $DP$ , in the planes  $BF$  and  $MN$ , are perpendicular to the intersection of these planes at the same point, the angle which they form is equal to the angle formed by the planes (D. 4); but this angle is a right angle: hence,  $BF$  is perpendicular to  $MN$ ; which was to be proved.



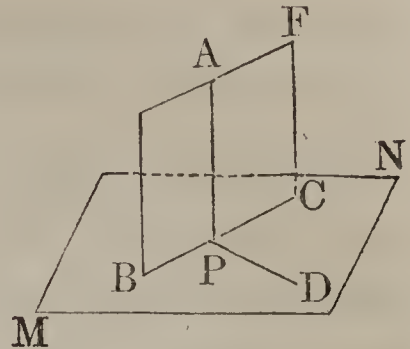
*Cor.* If three lines  $AP$ ,  $BP$ , and  $DP$ , are perpendicular to each other at a common point  $P$ , each line will be perpendicular to the plane of the other two, and the three planes will be perpendicular to each other.

## PROPOSITION XVII. THEOREM.

*If two planes are perpendicular to each other, a line drawn in one of them, perpendicular to their intersection, will be perpendicular to the other.*

Let the planes  $BF$  and  $MN$  be perpendicular to each other, and let the line  $AP$ , drawn in the plane  $BF$ , be perpendicular to the intersection  $BC$ ; then will  $AP$  be perpendicular to the plane  $MN$ .

For, in the plane  $MN$ , draw  $PD$  perpendicular to  $BC$  at  $P$ . Then because the planes  $BF$  and  $MN$  are perpendicular to each other, the angle  $APD$  will be a right angle: hence,  $AP$  is perpendicular to the two lines  $PD$  and  $BC$ , at their intersection, and consequently, is perpendicular to their plane  $MN$ ; *which was to be proved.*



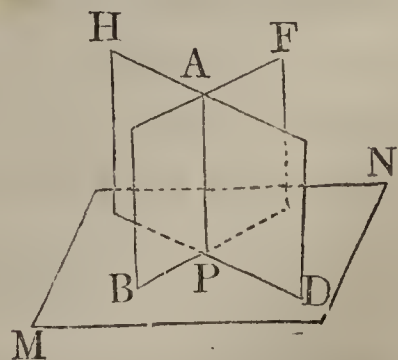
*Cor.* If the plane  $BF$  is perpendicular to the plane  $MN$ , and if at a point  $P$  of their intersection, we erect a perpendicular to the plane  $MN$ , that perpendicular will be in the plane  $BF$ . For, if not, draw in the plane  $BF$ ,  $PA$  perpendicular to  $PC$ , the common intersection;  $AP$  will be perpendicular to the plane  $MN$ , by the theorem; therefore, at the same point  $P$ , there are two perpendiculars to the plane  $MN$ ; which is impossible (P. IV., C. 2).

### PROPOSITION XVIII. THEOREM.

*If two planes cut each other, and are perpendicular to a third plane, their intersection is also perpendicular to that plane.*

Let the planes  $BF$ ,  $DH$ , be perpendicular to  $MN$ : then will their intersection  $AP$  be perpendicular to  $MN$ .

For, at the point  $P$ , erect a perpendicular to the plane  $MN$ ; that perpendicular must be in the plane  $BF$ , and also in the plane  $DH$  (P. XVII., C.); therefore, it is their common intersection  $AP$ : *which was to be proved.*



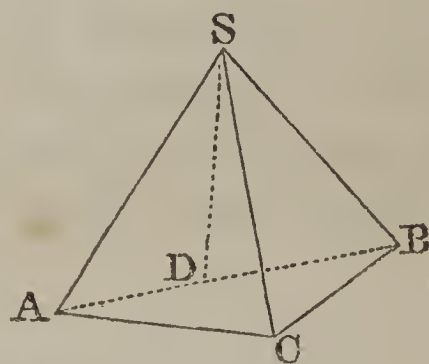
## PROPOSITION XIX. THEOREM.

*The sum of any two of the plane angles formed by the edges of a triedral angle, is greater than the third.*

Let  $SA$ ,  $SB$ , and  $SC$ , be the edges of a triedral angle: then will the sum of any two of the plane angles formed by them, as  $ASC$  and  $CSB$ , be greater than the third  $ASB$ .

If the plane angle  $ASB$  is equal to, or less than, either of the other two, the truth of the proposition is evident. Let us suppose, then, that  $ASB$  is greater than either.

In the plane  $ASB$ , construct the angle  $BSD$  equal to  $BSC$ ; draw  $AB$  in that plane, at pleasure; lay off  $SC$  equal to  $SD$ , and draw  $AC$  and  $CB$ . The triangles  $BSD$  and  $BSC$  have the side  $SC$  equal to  $SD$ , by construction, the side  $SB$  common, and the included angles  $BSD$  and  $BSC$  equal, by construction; the triangles are therefore equal in all their parts: hence,  $BD$  is equal to  $BC$ . But, from Proposition VII., Book I., we have,



$$BC + CA > BD + DA.$$

Taking away the equal parts  $BC$  and  $BD$ , we have,

$$CA > DA;$$

hence (B. I., P. IX., C.), we have,

$$\text{angle } ASC > \text{angle } ASD;$$

and, adding the equal angles  $BSC$  and  $BSD$ ,



angle  $ASC$  + angle  $CSB$  > angle  $ASD$  + angle  $DSB$  ;

or, angle  $ASC$  + angle  $CSB$  > angle  $ASB$  ;

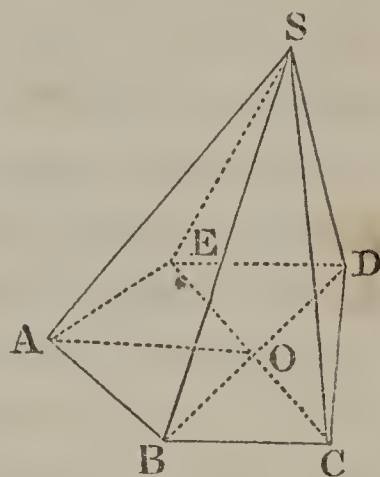
*which was to be proved.*

PROPOSITION XX. THEOREM.

*The sum of the plane angles formed by the edges of any polyedral angle, is less than four right angles.*

Let  $S$  be the vertex of any polyedral angle whose edges are,  $SA$ ,  $SB$ ,  $SC$ ,  $SD$ , and  $SE$  ; then will the sum of the angles about  $S$  be less than four right angles.

For, pass a plane cutting the edges in the points  $A$ ,  $B$ ,  $C$ ,  $D$ , and  $E$ , and the faces in the lines  $AB$ ,  $BC$ ,  $CD$ ,  $DE$ , and  $EA$ . From any point within the polygon thus formed, as  $O$ , draw the straight lines  $OA$ ,  $OB$ ,  $OC$ ,  $OD$ , and  $OE$ .

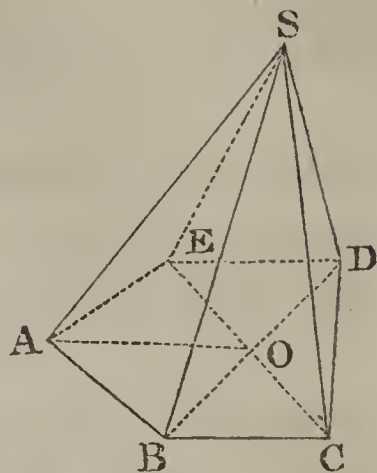


We then have two sets of triangles, one set having a common vertex  $S$ , the other having a common vertex  $O$ , and both having common bases  $AB$ ,  $BC$ ,  $CD$ ,  $DE$ ,  $EA$ . Now, in the set which has the common vertex  $S$ , the sum of all the angles is equal to the sum of all the plane angles formed by the edges of the polyedral angle whose vertex is  $S$ , together with the sum of all the angles at the bases : viz.,  $SAB$ ,  $SBA$ ,  $SBC$ , &c. ; and the entire sum is equal to twice as many right angles as there are triangles. In the set whose common vertex is  $O$ , the sum of all the angles is equal to the four right angles about  $O$ , together with the interior angles of the polygon, and this sum is equal to twice as many right angles as there are triangles. Since

the number of triangles, in each set, is the same, it follows that these sums are equal. But in the triedral angle whose vertex is  $B$ , we have (P. XIX.),

$$ABS + SBC > ABC;$$

and the like may be shown at each of the other vertices,  $C, D, E, A$ : hence, the sum of the angles at the bases, in the triangles whose common vertex is  $S$ , is greater than the sum of the angles at the bases, in the set whose common vertex is  $O$ : therefore, the sum of the vertical angles about  $S$ , is less than the sum of the angles about  $O$ : that is, less than four right angles; *which was to be proved.*



*Scholium.* The above demonstration is made on the supposition that the polyedral angle is convex, that is, that the dihedral angles of the consecutive faces are each less than two right angles.

#### PROPOSITION XXI. THEOREM.

*If the plane angles formed by the edges of two triedral angles are equal, each to each, the planes of the equal angles are equally inclined to each other.*

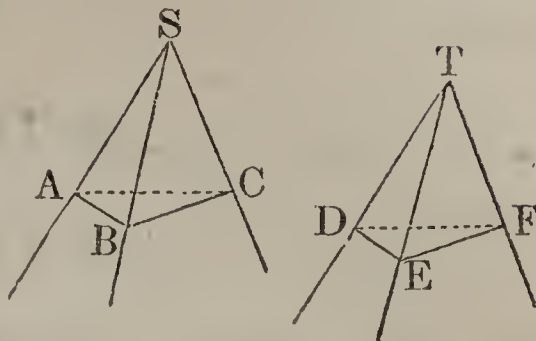
Let  $S$  and  $T$  be the vertices of two triedral angles, and let the angle  $ASC$  be equal to  $DTE$ ,  $ASB$  to  $DTE$ , and  $BSC$  to  $ETF$ : then will the planes of the equal angles be equally inclined to each other.

For, take any point of  $SB$ , as  $B$ , and from it draw in the two faces  $ASB$  and  $CSB$ , the lines  $BA$  and  $BC$ , respectively perpendicular to  $SB$ : then will the angle  $ABC$  measure the inclination of these faces. Lay off  $TE$  equal

to  $SB$ , and from  $E$  draw in the faces  $DTE$  and  $FTE$ , the lines  $ED$  and  $EF$ , respectively perpendicular to  $TE$ ; then will the angle  $DEF$

measure the inclination of these faces. Draw  $AC$  and  $DF$ .

The right-angled triangles  $SBA$  and  $TED$ , have the side  $SB$  equal to  $TE$ , and the angle  $ASB$  equal to  $DTE$ ; hence,  $AB$  is equal to  $DE$ , and  $AS$  to  $TD$ . In like manner, it may be shown that  $BC$  is equal to  $EF$ , and  $CS$  to  $FT$ . The triangles  $ASC$  and  $DTF$ , have the angle  $ASC$  equal to  $DTF$ , by hypothesis, the side  $AS$  equal to  $DT$ , and the side  $CS$  to  $FT$ , from what has just been shown; hence, the triangles are equal in all their parts, and consequently,  $AC$  is equal to  $DF$ . Now, the triangles  $ABC$  and  $DEF$  have their sides equal, each to each, and consequently, the corresponding angles are also equal; that is, the angle  $ABC$  is equal to  $DEF$ : hence, the inclination of the planes  $ASB$  and  $CSB$ , is equal to the inclination of the planes  $DTE$  and  $FTE$ . In like manner, it may be shown that the planes of the other equal angles are equally inclined; *which was to be proved.*



*Scholium.* If the planes of the equal plane angles are like placed, the triedral angles are equal in all respects, for they may be placed so as to coincide. If the planes of the equal angles are not similarly placed, the triedral angles are *equal by symmetry*. In this case, they may be placed so that two of the homologous faces shall coincide, the triedral angles lying on opposite sides of the plane, which is then called a *plane of symmetry*. In this position, for every point on one side of the plane of symmetry, there is a corresponding point on the other side.



## BOOK VII.

### P O L Y E D R O N S .

#### DEFINITIONS.

1. A POLYEDRON is a volume bounded by polygons.

The bounding polygons are called *faces* of the polyedron; the lines in which the polygons meet, are called *edges* of the polyedron; the points in which the edges meet, are called *vertices* of the polyedron.

2. A PRISM is a polyedron, two of whose faces are equal polygons having their homologous sides parallel, the other faces being parallelograms.

The equal polygons are called *bases* of the prism; one the *upper*, and the other the *lower base*; the parallelograms taken together make up the *lateral* or *convex surface* of the prism; the lines in which the lateral faces meet, are called *lateral edges* of the prism.



3. The ALTITUDE of a prism is the perpendicular distance between the planes of its bases.

4. A RIGHT PRISM is one whose lateral edges are perpendicular to the planes of the bases.

In this case, any lateral edge is equal to the altitude.



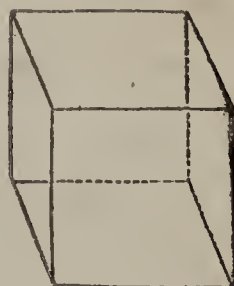
5. An **OBLIQUE PRISM** is one whose lateral edges are oblique to the planes of the bases.

In this case, any lateral edge is greater than the altitude.

6. Prisms are named from the number of sides of their bases; a *triangular prism* is one whose bases are triangles; a *quadrangular prism* is one whose bases are quadrilaterals; a *pentangular prism* is one whose bases are pentagons, and so on.

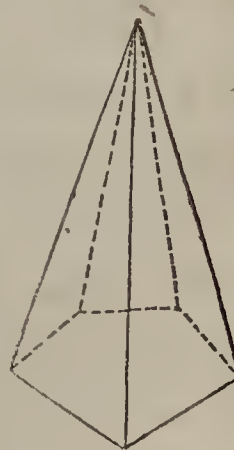
7. A **PARALLELOPIPEDON** is a prism whose bases are parallelograms.

A *Rectangular Parallelopipedon* is a right parallelopipedon, all of whose faces are rectangles; a *cube* is a rectangular parallelopipedon, all of whose faces are squares.



8. A **PYRAMID** is a polyedron bounded by a polygon called the *base*, and by triangles meeting at a common point, called the vertex of the pyramid.

The triangles taken together make up the *lateral or convex surface* of the pyramid; the lines in which the lateral faces meet, are called the lateral edges of the pyramid.



9. Pyramids are named from the number of sides of their bases; a *triangular pyramid* is one whose base is a triangle; a *quadrangular pyramid* is one whose base is a quadrilateral, and so on.

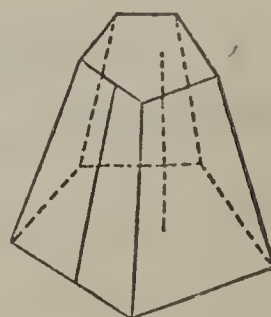
10. The **ALTITUDE** of a pyramid is the perpendicular distance from the vertex of the pyramid to the plane of its base.

11. A **RIGHT PYRAMID** is one whose base is a regular polygon, and in which the perpendicular drawn from the vertex to the plane of the base, passes through the centre of the base.

This perpendicular is called the axis of the pyramid.

12. The **SLANT HEIGHT** of a right pyramid, is the perpendicular distance from the vertex to any side of the base.

13. A **TRUNCATED PYRAMID** is that portion of a pyramid included between the base and any plane which cuts the pyramid.



When the cutting plane is parallel to the base, the truncated pyramid is called a **FRUSTUM OF A PYRAMID**, and the intersection of the cutting plane with the pyramid, is called the *upper base* of the frustum; the base of the pyramid is called the *lower base* of the frustum.

14. The **ALTITUDE** of a frustum of a pyramid, is the perpendicular distance between the planes of its bases.

15. The **SLANT HEIGHT** of a frustum of a right pyramid, is that portion of the slant height of the pyramid which lies between the planes of its upper and lower bases.

16. **SIMILAR POLYEDRONS** are those which are bounded by similar polygons, similarly placed.

Parts which are similarly placed, whether faces, edges, or angles, are called *homologous*.

17. A **DIAGONAL** of a polyedron, is a straight line joining the vertices of two polyedral angles not in the same face.



18. The VOLUME OF A POLYEDRON is its numerical value expressed in terms of some other polyedron as a unit.

The unit generally employed is a cube constructed on the linear unit as an edge.

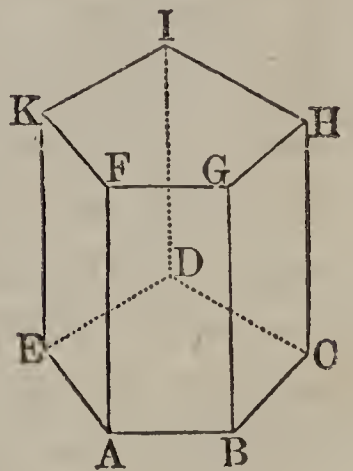
PROPOSITION I. THEOREM.

*The convex surface of a right prism is equal to the perimeter of either base multiplied by the altitude.*

Let  $ABCDE-K$  be a right prism: then is its convex surface equal to,

$$(AB + BC + CD + DE + EA) \times AF.$$

For, the convex surface is equal to the sum of all the rectangles  $AG$ ,  $BH$ ,  $CI$ ,  $DK$ ,  $EF$ , which compose it. Now, the altitude of each of the rectangles  $AF$ ,  $BG$ ,  $CH$ , &c., is equal to the altitude of the prism, and the area of each rectangle is equal to its base multiplied by its altitude (B. IV., P. V.): hence, the sum of these rectangles, or the convex surface of the prism, is equal to,



$$(AB + BC + CD + DE + EA) \times AF;$$

that is, to the perimeter of the base multiplied by the altitude; which was to be proved.

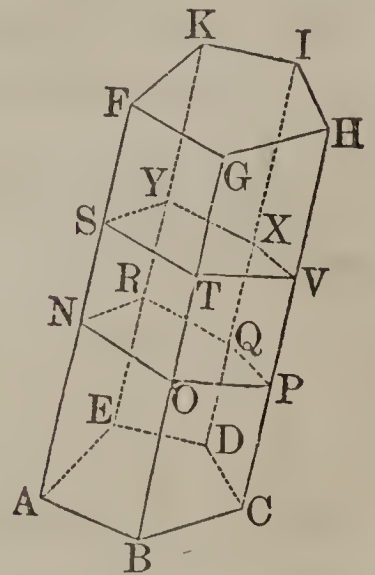
*Cor.* If two right prisms have the same altitude, their convex surfaces are to each other as the perimeters of their bases.

## PROPOSITION II. THEOREM.

*In any prism, the sections made by parallel planes are equal polygons.*

Let the prism  $AH$  be intersected by the parallel planes  $NP$ ,  $SV$ : then are the sections  $NOPQR$ ,  $STVXY$ , equal polygons.

For, the sides  $NO$ ,  $ST$ , are parallel, being the intersections of parallel planes with a third plane  $ABGF$ ; these sides,  $NO$ ,  $ST$ , are included between the parallels  $NS$ ,  $OT$ : hence,  $NO$  is equal to  $ST$  (B. I., P. XXVIII., C. 2). For like reasons, the sides  $OP$ ,  $PQ$ ,  $QR$ , &c., of  $NOPQR$ , are equal to the sides  $TV$ ,  $VX$ , &c., of  $STVXY$ , each to each; and since the equal sides are parallel, each to each, it follows that the angles  $NOP$ ,  $OPQ$ , &c., of the first section, are equal to the angles  $STV$ ,  $TVX$ , &c., of the second section, each to each (B. VI., P. XIII.): hence, the two sections  $NOPQR$ ,  $STVXY$ , are equal polygons; *which was to be proved.*



*Cor.* Every section of a prism, parallel to the bases, is equal to either base.

## PROPOSITION III. THEOREM.

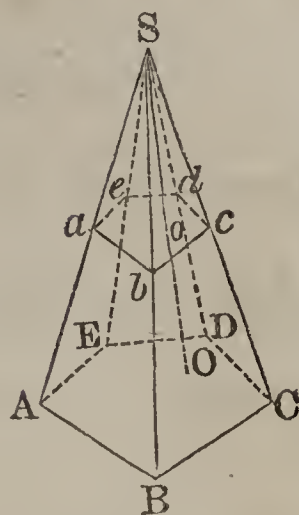
*If a pyramid be cut by a plane parallel to the base:*

- 1°. *The edges and the altitude will be divided proportionally:*
- 2°. *The section will be a polygon similar to the base.*

Let the pyramid  $S-ABCDE$ , whose altitude is  $SO$ , be cut by the plane  $abcde$ , parallel to the base  $ABCDE$ .

1°. The edges and altitude will be divided proportionally.

For, conceive a plane to be passed through the vertex  $S$ , parallel to the plane of the base; then will the edges and the altitude be cut by three parallel planes, and consequently they will be divided proportionally (B. VI., P. XV., C. 2); *which was to be proved.*



2°. The section  $abcde$ , will be similar to the base  $ABCDE$ . For,  $ab$  is parallel to  $AB$ , and  $bc$  to  $BC$  (B. VI., P. X.): hence, the angle  $abc$  is equal to the angle  $ABC$ . In like manner, it may be shown that each angle of the polygon  $abcde$  is equal to the corresponding angle of the base: hence, the two polygons are mutually equiangular.

Again, because  $ab$  is parallel to  $AB$ , we have,

$$ab : AB :: sb : SB ;$$

and, because  $bc$  is parallel to  $BC$ , we have,

$$bc : BC :: sb : SB ;$$

hence (B. II., P. IV.), we have,

$$ab : AB :: bc : BC.$$

In like manner, it may be shown that all the sides of  $abcde$  are proportional to the corresponding sides of the polygon  $ABCDE$ : hence, the section  $abcde$  is similar to the base  $ABCDE$  (B. IV., D. 1); *which was to be proved.*

*Cor. 1.* If two pyramids  $S-ABCDE$ , and  $S-XYZ$ , having a common vertex  $S$ , and their bases in the same plane, be cut by a plane  $abc$ , parallel to the plane of their bases, the sections will be to each other as the bases.



For, the polygons  $abcd$  and  $ABCD$ , being similar, are to each other as the squares of their homologous sides  $ab$  and  $AB$  (B. IV., P. XXVII); but,

$$\overline{ab}^2 : \overline{AB}^2 :: \overline{Sa}^2 : \overline{SA}^2 : \overline{So}^2 : \overline{SO}^2;$$

hence (B. II., P. IV.), we have,

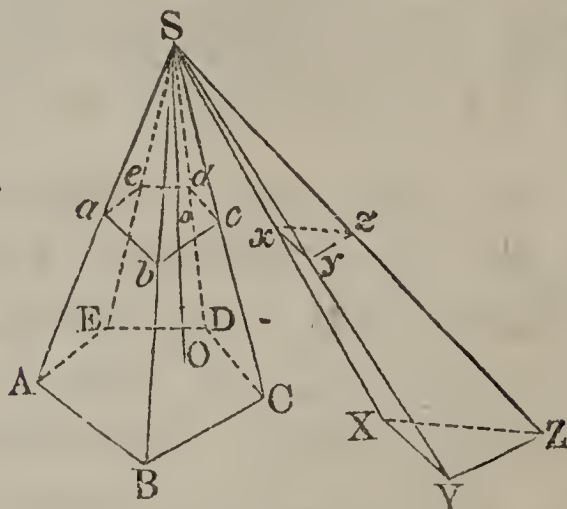
$$abcde : ABCDE :: \overline{So}^2 : \overline{SO}^2.$$

In like manner, we have,

$$xyz : XYZ :: \overline{So}^2 : \overline{SO}^2;$$

hence,

$$abcde : ABCDE :: xyz : XYZ.$$



*Cor. 2.* If the bases are equal, any sections at equal distances from the bases will be equal.

*Cor. 3.* The area of any section parallel to the base, is proportional to the square of its distance from the vertex.

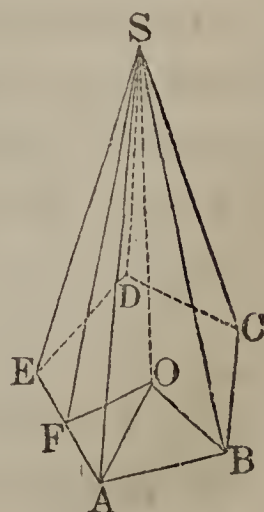
#### PROPOSITION IV. THEOREM.

*The convex surface of a right pyramid is equal to the perimeter of its base multiplied by half the slant height.*

Let  $S$  be the vertex,  $ABCDE$  the base, and  $SF$ , perpendicular to  $EA$ , the slant height of a right pyramid: then will the convex surface be equal to,

$$(AB + BC + CD + DE + EA) \times \frac{1}{2}SF.$$

Draw  $SO$  perpendicular to the plane of the base.



From the definition of a right pyramid, the point  $O$  is the centre of the base (D. 11) : hence, the lateral edges,  $SA$ ,  $SB$ , &c., are all equal (B. VI., P. V.) ; but the sides of the base are all equal, being sides of a regular polygon : hence, the lateral faces are all equal, and consequently their altitudes are all equal, each being equal to the slant height of the pyramid.

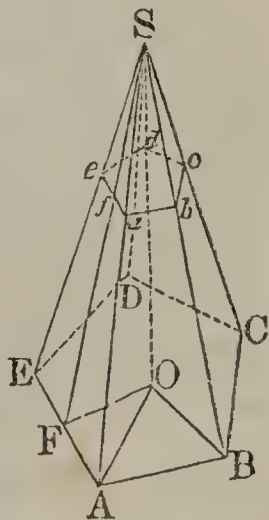
Now, the area of any lateral face, as  $SEA$ , is equal to its base  $EA$ , multiplied by half its altitude  $SF$  : hence, the sum of the areas of the lateral faces, or the convex surface of the pyramid, is equal to,

$$(AB + BC + CD + DE + EA) \times \frac{1}{2}SF ;$$

*which was to be proved.*

*Scholium.* The convex surface of a frustum of a right pyramid is equal to half the sum of the perimeters of its upper and lower bases, multiplied by the slant height.

Let  $ABCDE-e$  be a frustum of a right pyramid, whose vertex is  $S$  : then will the section  $abcde$  be similar to the base  $ABCDE$ , and their homologous sides will be parallel, (P. III.). Any lateral face of the frustum, as  $AEea$ , is a trapezoid, whose altitude is equal to  $Ff$ , the slant height of the frustum ; hence, its area is equal to  $\frac{1}{2}(EA + ea) \times Ff$  (B. IV., P. VII.). But the area of the con-



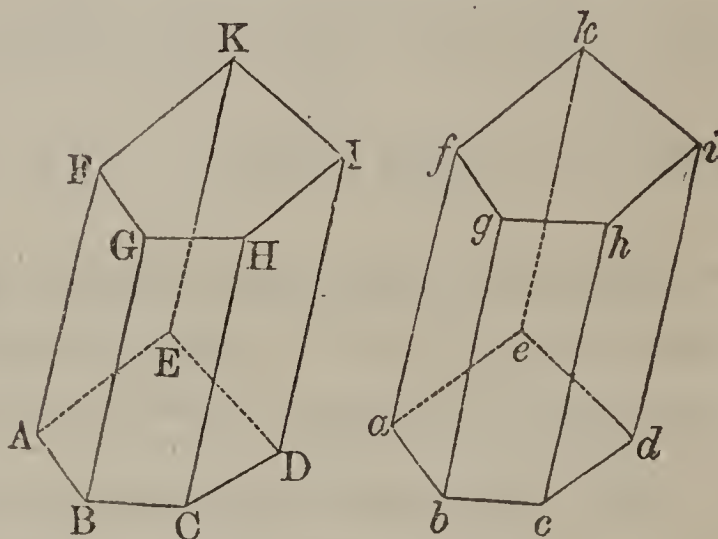
vex surface of the frustum is equal to the sum of the areas of its lateral faces ; it is, therefore, equal to the half sum of the perimeters of its upper and lower bases, multiplied by half the slant height.

## PROPOSITION V. THEOREM.

*If the three faces which include a triedral angle of a prism are equal to the three faces which include a triedral angle of a second prism, each to each, and are like placed, the two prisms are equal in all their parts.*

Let  $B$  and  $b$  be the vertices of two triedral angles, included by faces respectively equal to each other, and similarly placed: then will the prism  $ABCDE-K$  be equal to the prism  $abcde-k$ , in all of its parts.

For, place the base  $abcde$  upon the equal base  $ABCDE$ , so that they shall coincide; then because the triedral angles whose vertices are  $b$  and  $B$ , are equal, the parallelogram  $bh$  will coincide with  $BH$ , and the parallelogram  $bf$  with  $BF$ : hence, the two



sides  $fg$  and  $gh$ , of one upper base, will coincide with the homologous sides of the other upper base; and because the upper bases are equal, they must coincide throughout; consequently, each of the lateral faces of one prism will coincide with the corresponding lateral face of the other prism: the prisms, therefore, coincide throughout, and are therefore equal in all their parts; *which was to be proved.*

*Cor.* If two right prisms have their bases equal in all their parts, and have also equal altitudes, the prisms themselves will be equal in all their parts. For, the faces which include any triedral angle of the one, will be equal to the faces which include the corresponding triedral angle of the other, each to each, and they will be similarly placed.

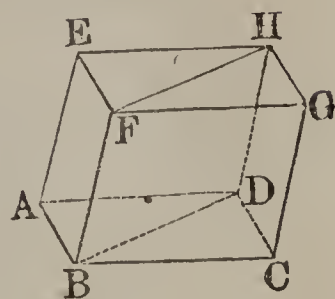


## PROPOSITION VI. THEOREM.

*In any parallelopipedon, the opposite faces are equal, each to each, and their planes are parallel.*

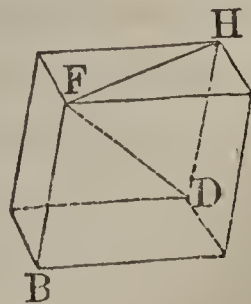
Let  $ABCD-H$  be a parallelopipedon : then will its opposite faces be equal and their planes will be parallel.

For, the bases,  $ABCD$  and  $EFGH$  are equal, and their planes parallel by definition (D. 7). The opposite faces  $AEHD$  and  $BFGC$ , have the sides  $AE$  and  $BF$  parallel, because they are opposite sides of the parallelogram  $BE$ ; and the sides  $EH$  and  $FG$  parallel, because they are opposite sides of the parallelogram  $EG$ ; and consequently, the angles  $AEH$  and  $BFG$  are equal (B. VI., P. XIII.). But the side  $AE$  is equal to  $BF$ , and the side  $EH$  to  $FG$ ; hence, the faces  $AEHD$  and  $BFGC$  are equal; and because  $AE$  is parallel to  $BF$ , and  $EH$  to  $FG$ , the planes of the faces are parallel (B. VI., P. XIII.). In like manner, it may be shown that the parallelograms  $ABFE$  and  $DCGH$ , are equal and their planes parallel: hence, the opposite faces are equal, each to each, and their planes are parallel; *which was to be proved.*



*Cor. 1.* Any two opposite faces of a parallelopipedon may be taken as bases.

*Cor. 2.* In a rectangular parallelopipedon, the square of either of the diagonals is equal to the sum of the squares of the three edges which meet at the same vertex.



For, let  $FD$  be either of the diagonals, and draw  $FH$ .

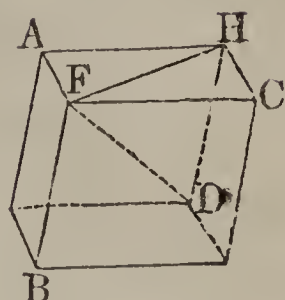
Then, in the right-angled triangle  $FHD$ , we have,

$$\overline{FD}^2 = \overline{DH}^2 + \overline{FH}^2.$$

But  $DH$  is equal to  $FB$ , and  $\overline{FH}^2$  is equal to  $\overline{FA}^2$  plus  $\overline{AH}^2$  or  $\overline{FC}^2$ :

hence,

$$\overline{FD}^2 = \overline{FB}^2 + \overline{FA}^2 + \overline{FC}^2.$$



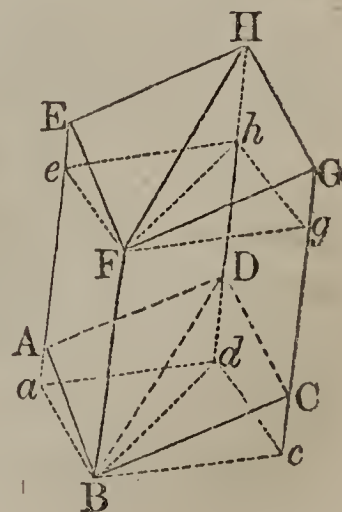
*Cor. 3.* A parallelepipedon may be constructed on three lines  $AB$ ,  $AD$ , and  $AE$ , intersecting in a common point  $A$ , and not lying in the same plane. For, pass through the extremity of each line, a plane parallel to the plane of the other two lines; then will these planes, together with the planes of the given lines, determine a parallelepipedon.

#### PROPOSITION VII. THEOREM.

*If a plane be passed through the diagonally opposite edges of a parallelepipedon, it will divide the parallelepipedon into two equal triangular prisms.*

Let  $ABCD-H$  be a parallelepipedon, and let a plane be passed through the edges  $BF$  and  $DH$ : then will the prisms  $ABD-H$  and  $BCD-H$  be equal in volume.

For, through the vertices  $F$  and  $B$  let planes be passed perpendicular to  $FB$ , the former cutting the other lateral edges in the points  $e$ ,  $h$ ,  $g$ , and the latter cutting those edges produced, in the points  $a$ ,  $d$ , and  $c$ . The sections  $Fehg$  and  $Badc$  will be parallelograms,



because their opposite sides are parallel, each to each (B. VI., P. X.) ; they will also be equal (P. II.) : hence, the polyedron  $Badc-g$  is a right prism (D. 2, 4), as are also the polyedrons  $Bad-h$  and  $Bcd-h$ .

Place the triangle  $Feh$  upon  $Bad$ , so that  $F$  shall coincide with  $B$ ,  $e$  with  $a$ , and  $h$  with  $d$  ; then, because  $eE$ ,  $hH$ , are perpendicular to the plane  $Feh$ , and  $aA$ ,  $dD$ , to the plane  $Bad$ , the line  $eE$  will take the direction  $aA$ , and the line  $hH$  the direction  $dD$ . The lines  $AE$  and  $ae$  are equal, because each is equal to  $BF$  (B. I., P. XXVIII.). If we take away from the line  $aE$  the part  $ae$ , there will remain the part  $eE$  ; and if from the same line, we take away the part  $AE$ , there will remain the part  $Aa$  : hence,  $eE$  and  $aA$  are equal (A. 3) ; for a like reason  $hH$  is equal to  $dD$  : hence, the point  $E$  will coincide with  $A$ , and the point  $H$  with  $D$ , and consequently, the polyedrons  $Feh-H$  and  $Bad-D$  will coincide throughout, and are therefore equal.

If from the polyedron  $Bad-H$ , we take away the part  $Bad-D$ , there will remain the prism  $BAD-H$  ; and if from the same polyedron we take away the part  $Feh-H$ , there will remain the prism  $Bad-h$  : hence, these prisms are equal in volume. In like manner, it may be shown that the prisms  $BCD-H$  and  $Bcd-h$  are equal in volume.

The prisms  $Bad-h$ , and  $Bcd-h$ , have equal bases, because these bases are halves of equal parallelograms (B. I., P. XXVIII., C. 1) ; they have also equal altitudes ; they are therefore equal (P. V., C.) : hence, the prisms  $BAD-H$  and  $BCD-H$  are equal (A. 1) ; *which was to be proved.*

*Cor.* Any triangular prism  $ABD-H$ , is equal to half of the parallelopipedon  $AG$ , which has the same triedral angle  $A$ , and the same edges  $AB$ ,  $AD$ , and  $AE$ .

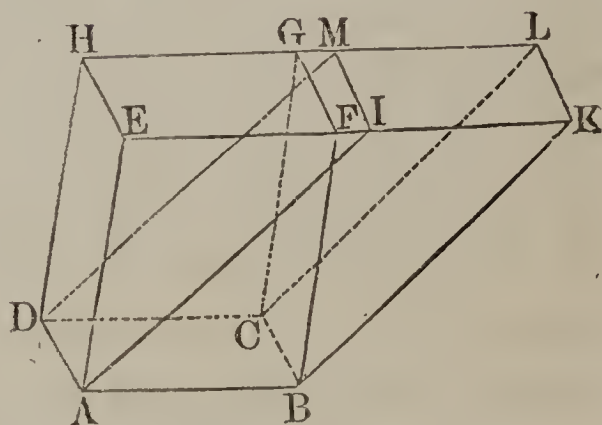


## PROPOSITION VIII. THEOREM.

*If two parallelopipedons have a common lower base, and their upper bases between the same parallels, they are equal in volume.*

Let the parallelopipedons  $AG$  and  $AL$  have the common lower base  $ABCD$ , and their upper bases  $EFGH$  and  $IKLM$ , between the same parallels  $EK$  and  $HL$ : then will they be equal in volume.

For, the lines  $EF$  and  $IK$  are equal, because each is equal to  $AB$ ; hence, the sum of  $EF$  and  $FI$ , or  $EI$ , is equal to the sum of  $FI$  and  $IK$ , or  $FK$ . In the triangular prisms  $AEI-M$  and  $BFK-L$ , we have the line  $AE$  equal and parallel to  $BF$ , and  $EI$  equal to  $FK$ ; hence, the face  $AEI$  is equal to  $BFK$ . The line  $EH$  is equal and parallel to  $FG$ , and  $EI$  is equal and parallel to  $FK$ ; hence, the face  $EIMH$  is equal to  $FKLG$ : the faces  $AEHD$  and  $BFGC$  are also equal (P. VI.): hence, the prisms are equal (P. V.).



If from the polyedron  $ABKE-H$ , we take away the prism  $BFK-L$ , there will remain the parallelopipedon  $AG$ ; and if from the same polyedron we take away the prism  $AEI-M$ , there will remain the parallelopipedon  $AL$ : hence, these parallelopipedons are equal in volume (A. 3); *which was to be proved.*

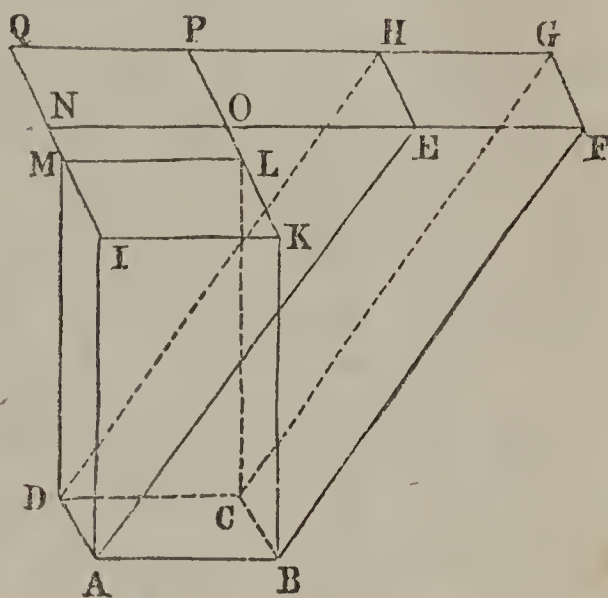
## PROPOSITION IX. THEOREM.

*If two parallelopipedons have a common lower base and the same altitude, they will be equal in volume.*

Let the parallelopipedons  $AG$  and  $AL$  have the common lower base  $ABCD$  and the same altitude: then will they be equal in volume.

Because they have the same altitude, their upper bases will lie in the same plane.

Let the sides  $IM$  and  $KL$  be prolonged, and also the sides  $FE$  and  $GH$ ; these prolongations will form a parallelogram  $OQ$ , which will be equal to the common base of the given parallelopipedons, because its sides are respectively parallel and equal to the corresponding sides of that base.



Now, if a third parallelopipedon be constructed, having for its lower base the parallelogram  $ABCD$ , and for its upper base  $NOPQ$ , this third parallelopipedon will be equal in volume to the parallelopipedon  $AG$ , since they have the same lower base, and their upper bases between the same parallels,  $QG$ ,  $NF$  (P. VIII.). For a like reason, this third parallelopipedon will also be equal in volume to the parallelopipedon  $AL$ : hence, the two parallelopipedons  $AG$ ,  $AL$ , are equal in volume; *which was to be proved.*

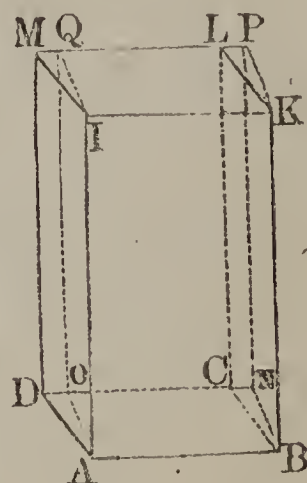
*Cor.* Any oblique parallelopipedon is equal in volume to a right parallelopipedon, having the same base and an equal altitude.

## PROPOSITION X. PROBLEM.

*To construct a rectangular parallelepipedon which shall be equal in volume to a right parallelepipedon whose base is any parallelogram.*

Let  $ABCD-M$  be a right parallelepipedon, having for its base the parallelogram  $ABCD$ .

Through the edges  $AI$  and  $BK$  pass the planes  $AQ$  and  $BP$ , respectively perpendicular to the plane  $AK$ , the former meeting the face  $DL$  in  $OQ$ , and the latter meeting that face produced in  $NP$ : then will the polyedron  $AP$  be a rectangular parallelepipedon equal to the given parallelepipedon. It will be a rectangular parallelepipedon, because all of its faces are rectangles, and it will be equal to the given parallelepipedon, because the two may be regarded as having the common base  $AK$  (P. VI., C. 1), and an equal altitude  $AO$  (P. IX.).



*Cor. 1.* A right parallelepipedon, whose base is any parallelogram, is equal in volume to a rectangular parallelepipedon having an equal base and the same altitude. For, the base  $AN$  is equal to the base  $AC$  (B. IV., P. I.); and the altitude  $AI$  is common.

*Cor. 2.* An oblique parallelepipedon is equal in volume to a rectangular parallelepipedon, having an equal base and an equal altitude.

*Cor. 3.* Any two parallelepipedons are equal in volume, when they have equal bases and equal altitudes.



PROPOSITION XI. THEOREM.

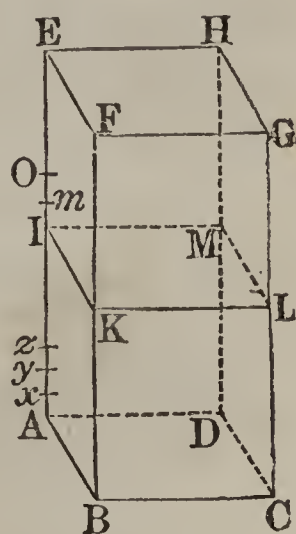
*Two rectangular parallelopipedons having a common lower base, are to each other as their altitudes.*

Let the parallelopipedons  $AG$  and  $AL$  have the common lower base  $ABCD$ : then will they be to each other as their altitudes  $AE$  and  $AI$ .

1°. Let the altitudes be commensurable, and suppose, for example, that  $AE$  is to  $AI$ , as 15 is to 8.

Conceive  $AE$  to be divided into 15 equal parts, of which  $AI$  will contain 8; through the points of division let planes be passed parallel to  $ABCD$ . These planes will divide the parallelopipedon  $AG$  into 15 parallelopipedons, which have equal bases (P. II. C.) and equal altitudes; hence, they are equal (P. X., Cor. 3).

Now,  $AG$  contains 15, and  $AL$  8 of these equal parallelopipedons; hence,  $AG$  is to  $AL$ , as 15 is to 8, or as  $AE$  is to  $AI$ . In like manner, it may be shown that  $AG$  is to  $AL$ , as  $AE$  is to  $AI$ , when the altitudes are to each other as any other whole numbers.



2°. Let the altitudes be incommensurable.

Now, if  $AG$  is not to  $AL$ , as  $AE$  is to  $AI$ , let us suppose that,

$$AG : AL :: AE : AO,$$

in which  $AO$  is greater than  $AI$ .

Divide  $AE$  into equal parts, such that each shall be less than  $OI$ ; there will be at least one point of division

$m$ , between  $O$  and  $I$ . Let  $P$  denote the parallelopipedon, whose base is  $ABCD$ , and altitude  $Am$ ; since the altitudes  $AE$ ,  $Am$ , are to each other as two whole numbers, we have,

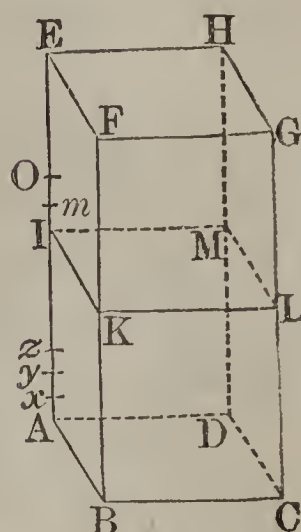
$$AG : P :: AE : Am.$$

But, by hypothesis, we have,

$$AG : AL :: AE : AO;$$

therefore (B. II., P. IV., C.),

$$AL : P :: AO : Am.$$



But  $AO$  is greater than  $Am$ ; hence, if the proportion is true,  $AL$  must be greater than  $P$ . On the contrary, it is less; consequently, the fourth term of the proportion cannot be greater than  $AI$ . In like manner, it may be shown that the fourth term cannot be less than  $AI$ ; it is, therefore, equal to  $AI$ . In this case, therefore,  $AG$  is to  $AL$ , as  $AE$  is to  $AI$ .

Hence, in all cases, the given parallelopipedons are to each other as their altitudes; *which was to be proved*.

*Scholium.* Any two rectangular parallelopipedons whose bases are equal, are to each other as their altitudes.

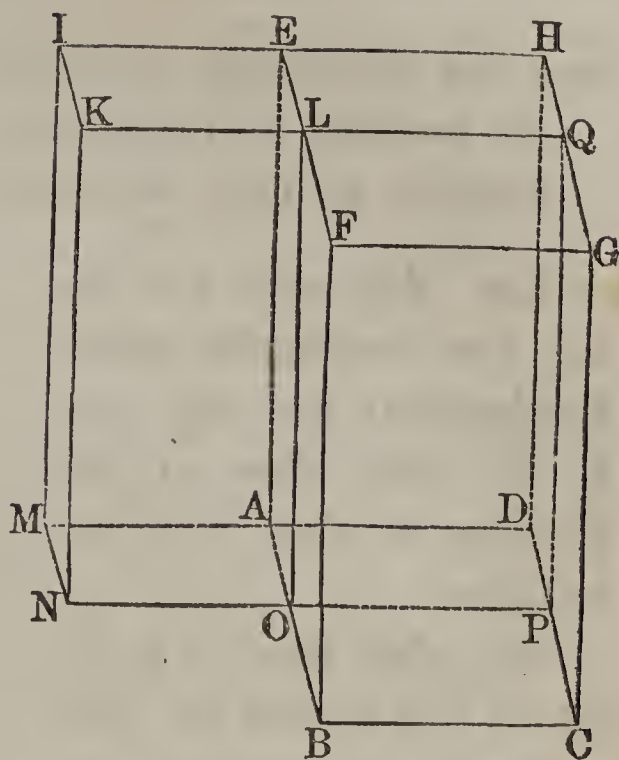
#### PROPOSITION XII. THEOREM.

*Two rectangular parallelopipedons having equal altitudes, are to each other as their bases.*

Let the rectangular parallelopipedons  $AG$  and  $AK$  have the same altitude  $AE$ : then will they be to each other as their bases.

For, place them as shown in the figure, and produce the plane of the face  $NL$ , until it intersects the plane of the face  $HC$ , in  $PQ$ ; we shall thus form a third rectangular parallelopipedon  $AQ$ .

The parallelopipedons  $AG$  and  $AQ$  have a common base  $AH$ ; they are therefore to each other as their altitudes  $AB$  and  $AO$  (P. XI.): hence, we have the proportion,



$$\text{vol. } AG : \text{vol. } AQ :: AB : AO.$$

The parallelopipedons  $AQ$  and  $AK$  have the common base  $AL$ ; they are therefore to each other as their altitudes  $AD$  and  $AM$ : hence,

$$\text{vol. } AQ : \text{vol. } AK :: AD : AM.$$

Multiplying these proportions, term by term (B. II., P. XII.), and omitting the common factor,  $\text{vol. } AQ$ , we have,

$$\text{vol. } AG : \text{vol. } AK :: AB \times AD : AO \times AM.$$

But  $AB \times AD$  is equal to the area of the base  $ABCD$  and  $AO \times AM$  is equal to the area of the base  $AMNO$ : hence, two rectangular parallelopipedons having equal altitudes, are to each other as their bases; *which was to be proved.*



## PROPOSITION XIII. THEOREM.

*Any two rectangular parallelopipedons are to each other as the products of their bases and altitudes ; that is, as the products of their three dimensions.*

Let  $AZ$  and  $AG$  be any two rectangular parallelopipedons: then will they be to each other as the products of their three dimensions.

For, place them as in the figure, and produce the faces necessary to complete the rectangular parallelopipedon  $AK$ . The parallelopipedons  $AZ$  and  $AK$  have a common base  $AN$ ; hence (P. XI.),

$$\text{vol. } AZ : \text{vol. } AK :: AX : AE.$$

The parallelopipedons  $AK$  and  $AG$  have a common altitude  $AE$ ; hence (P. XII.),

$$\text{vol. } AK : \text{vol. } AG :: AMNO : ABCD.$$

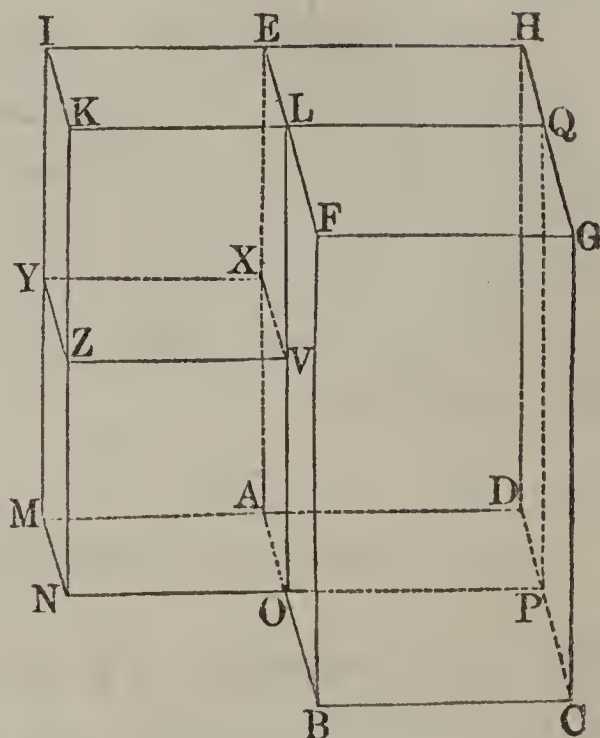
Multiplying these proportions, term by term, and omitting the common factor,  $\text{vol. } AK$ , we have,

$$\text{vol. } AZ : \text{vol. } AG :: AMNO \times AX : ABCD \times AE;$$

or, since  $AMNO$  is equal to  $AM \times AO$ , and  $ABCD$  to  $AB \times AD$ ,

$$\text{vol. } AZ : \text{vol. } AG :: AM \times AO \times AX : AB \times AD \times AE;$$

*which was to be proved.*



*Cor. 1.* If we make the three edges  $AM$ ,  $AO$ , and  $AX$ , each equal to the linear unit, the parallelopipedon  $AZ$  will be a cube constructed on that unit, as an edge; and consequently, it will be the unit of volume. Under this supposition, the last proposition becomes,

$1 : \text{vol. } AG :: 1 : AB \times AD \times AE;$   
whence,

$$\text{vol. } AG = AB \times AD \times AE.$$

Hence, *the volume of any rectangular parallelopipedon is equal to the product of its three dimensions*; that is, the number of times which it contains the unit of volume, is equal to the number of linear units in its length, by the number of linear units in its breadth, by the number of linear units in its height.

*Cor. 2.* *The volume of a rectangular parallelopipedon is equal to the product of its base and altitude*; that is, the number of times which it contains the unit of volume, is equal to the number of superficial units in its base, multiplied by the number of linear units in its altitude.

*Cor. 3.* The volume of any parallelopipedon is equal to the product of its base and altitude (P. X., C. 2).

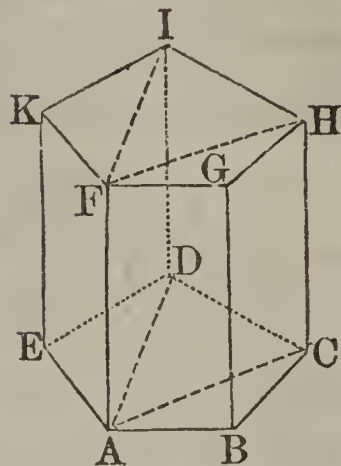
#### PROPOSITION XIV. THEOREM.

*The volume of any prism is equal to the product of its base and altitude.*

Let  $ABCDE-K$  be any prism: then is its volume equal to the product of its base and altitude.

For, through any lateral edge, as  $AF$ , pass the planes  $AH$ ,  $AI$ , dividing it into triangular prisms. These prisms will all have a common altitude equal to that of the given prism.

Now, the volume of any one of the triangular prisms, as  $ABC-H$ , is equal to half that of a parallelopipedon constructed on the edges  $BA$ ,  $BC$ ,  $BG$  (P. VII., C.); but the volume of this parallelopipedon is equal to the product of its base and altitude (P. XIII., C. 3); and because the base of the prism is half that of the parallelopipedon, the volume of the prism is also equal to the product of its base and altitude: hence, the sum of the triangular prisms, which make up the given prism, is equal to the sum of their bases, which make up the base of the given prism, into their common altitude; *which was to be proved.*



*Cor.* Any two prisms are to each other as the products of their bases and altitudes. Prisms having equal bases are to each other as their altitudes. Prisms having equal altitudes are to each other as their bases.

#### PROPOSITION XV. THEOREM.

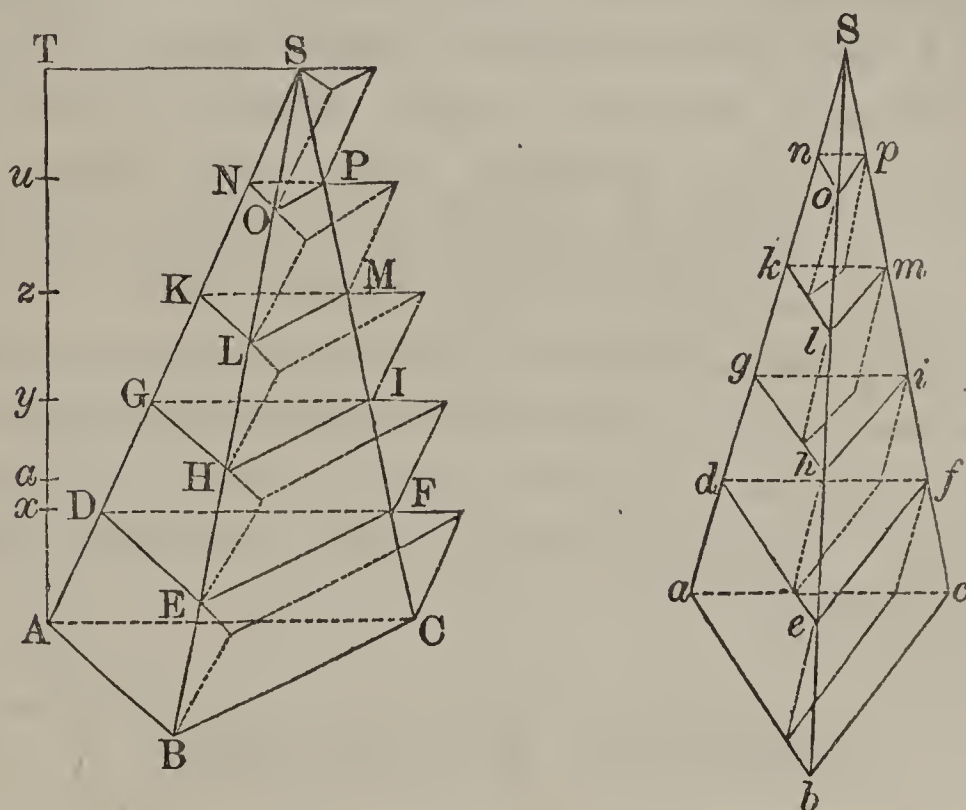
*Two triangular pyramids having equal bases and equal altitudes, are equal in volume.*

Let  $S-ABC$ , and  $S-abc$ , be two pyramids having their equal bases  $ABC$  and  $abc$  in the same plane, and let  $AT$  be their common altitude: then will they be equal in volume.

For, if they are not equal in volume, suppose one of them, as  $S-ABC$ , to be the greater, and let their difference be equal to a prism whose base is  $ABC$ , and whose altitude is  $Aa$ .



Divide the altitude  $AT$  into equal parts  $Ax$ ,  $xy$ , &c., each of which is less than  $Aa$ , and let  $k$  denote one of these parts; through the points of division pass planes parallel to the plane of the bases; the sections of the two pyramids, by each of these planes, will be equal, namely,  $DEF$  to  $def$ ,  $GHI$  to  $ghi$ , &c. (P. III., C. 2).



On the triangles  $ABC$ ,  $DEF$ , &c., taken as lower bases, construct exterior prisms whose edges shall be parallel to  $AS$ , and whose altitudes shall be equal to  $k$ : and on the triangles  $def$ ,  $ghi$ , &c., taken as upper bases, construct interior prisms, whose edges shall be parallel to  $Sa$ , and whose altitudes shall be equal to  $k$ . It is evident that the sum of the exterior prisms is greater than the pyramid  $S-ABC$ , and also that the sum of the interior prisms is less than the pyramid  $S-abc$ : hence, the difference between the sum of the exterior and the sum of the interior prisms, is greater than the difference between the two pyramids.

Now, beginning at the bases, the second exterior prism  $EFD-G$ , is equal to the first interior prism  $efd-a$ ,

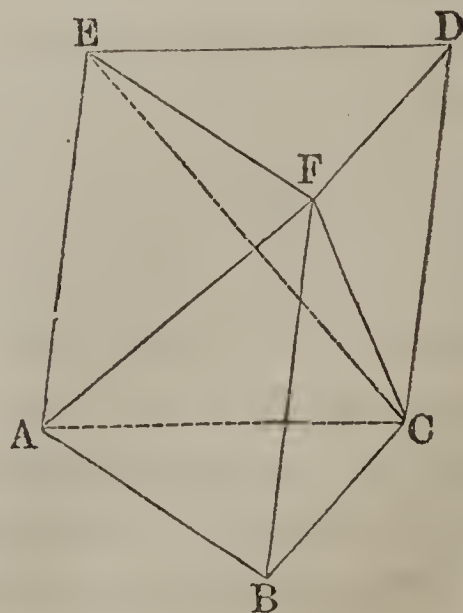
because they have the same altitude  $k$ , and their bases  $EFD$ ,  $efd$ , are equal: for a like reason, the third exterior prism  $HIG-K$ , and the second interior prism  $hig-d$ , are equal, and so on to the last in each set: hence, each of the exterior prisms, excepting the first  $BCA-D$ , has an equal corresponding interior prism; the prism  $BCA-D$ , is, therefore, the difference between the sum of all the exterior prisms, and the sum of all the interior prisms. But the difference between these two sets of prisms is greater than that between the two pyramids, which latter difference was supposed to be equal to a prism whose base is  $BCA$ , and whose altitude is equal to  $Aa$ , greater than  $k$ ; consequently, the prism  $BCA-D$  is greater than a prism having the same base and a greater altitude, which is impossible. hence, the supposed inequality between the two pyramids cannot exist; they are, therefore, equal in volume; *which was to be proved.*

PROPOSITION XVI. THEOREM.

*Any triangular prism may be divided into three triangular pyramids, equal to each other in volume.*

Let  $ABC-D$  be a triangular prism: then can it be divided into three equal triangular pyramids.

For, through the edge  $AC$ , pass the plane  $ACE$ , and through the edge  $EF$  pass the plane  $EFC$ . The pyramids  $ACE-F$  and  $ECD-F$ , have their bases  $ACE$  and  $ECD$  equal, because they are halves of the same parallelogram  $ACDE$ ; and they have a common



altitude, because their bases are in the same plane  $AD$ , and their vertices at the same point  $F$ ; hence, they are equal in volume (P. XV.). The pyramids  $ABC-F$  and  $DEF-C$ , have their bases  $ABC$  and  $DEF$ , equal, because they are the bases of the given prism, and their altitudes are equal because each is equal to the altitude of the prism; they are, therefore, equal in volume: hence, the three pyramids into which the prism is divided, are all equal in volume; *which was to be proved.*

*Cor. 1.* A triangular pyramid is one-third of a prism, having an equal base and an equal altitude.

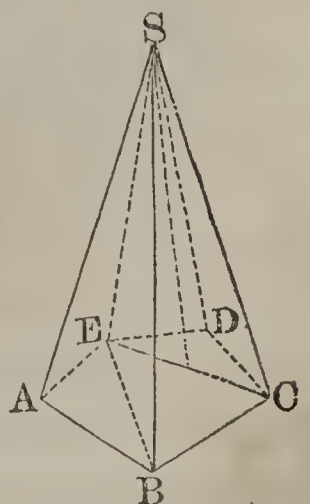
*Cor. 2.* The volume of a triangular pyramid is equal to one-third of the product of its base and altitude.

#### PROPOSITION XVII. THEOREM.

*The volume of any pyramid is equal to one-third of the product of its base and altitude.*

Let  $S-ABCDE$ , be any pyramid: then is its volume equal to one-third of the product of its base and altitude.

For, through any lateral edge, as  $SE$ , pass the planes  $SEB$ ,  $SEC$ , dividing the pyramid into triangular pyramids. The altitudes of these pyramids will be equal to each other, because each is equal to that of the given pyramid. Now, the volume of each triangular pyramid is equal to one-third of the product of its base and altitude (P. XVI., C. 2); hence, the sum of the volumes of the triangular pyramids, is equal to one-third of the product of the sum of their bases





by their common altitude. But the sum of the triangular pyramids is equal to the given pyramid, and the sum of their bases is equal to the base of the given pyramid: hence, the volume of the given pyramid is equal to one-third of the product of its base and altitude; *which was to be proved.*

*Cor. 1.* The volume of a pyramid is equal to one-third of the volume of a prism having an equal base and an equal altitude.

*Cor. 2.* Any two pyramids are to each other as the products of their bases and altitudes. Pyramids having equal bases are to each other as their altitudes. Pyramids having equal altitudes are to each other as their bases.

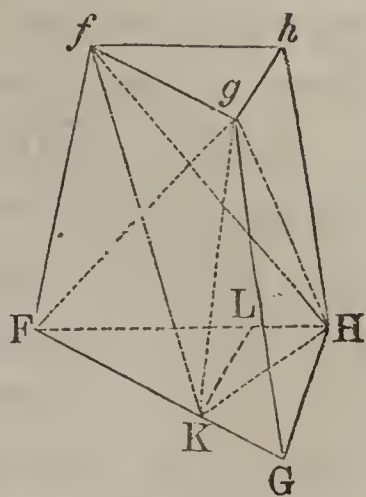
*Scholium.* The volume of a polyedron may be found by dividing it into triangular pyramids, and computing their volumes separately. The sum of these volumes will be equal to the volume of the polyedron.

#### PROPOSITION XVIII. THEOREM.

*The volume of a frustum of any triangular pyramid is equal to the sum of the volumes of three pyramids whose common altitude is that of the frustum, and whose bases are the lower base of the frustum, the upper base of the frustum, and a mean proportional between the two bases.*

Let  $FGH-h$  be a frustum of any triangular pyramid: then will its volume be equal to that of three pyramids whose common altitude is that of the frustum, and whose bases are the lower base  $FGH$ , the upper base  $fg h$ , and a mean proportional between their bases.

For, through the edge  $FH$ , pass the plane  $FHg$ , and through the edge  $fg$ , pass the plane  $fgH$ , dividing the frustum into three pyramids. The pyramid  $g-FGH$ , has for its base the lower base  $FGH$  of the frustum, and its altitude is equal to that of the frustum, because its vertex  $g$ , is in the plane of the upper base. The pyramid  $H-fgh$ , has for its base the upper base  $fgh$  of the frustum, and its altitude is equal to that of the frustum, because its vertex lies in the plane of the lower base.



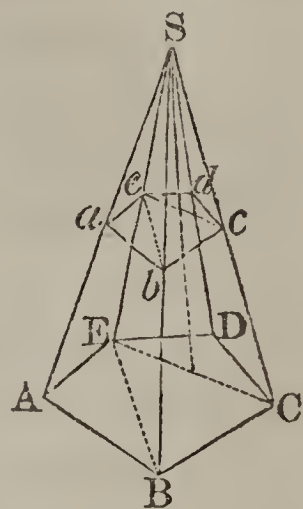
The remaining pyramid may be regarded as having the triangle  $FfH$  for its base, and the point  $g$  for its vertex. From  $g$ , draw  $gK$  parallel to  $fF$ , and draw also  $KH$  and  $Kf$ . Then will the pyramids  $K-FfH$  and  $g-FfH$ , be equal; for they have a common base, and their altitudes are equal, because their vertices  $K$  and  $g$  are in a line parallel to the base (B. VI., P. XII., C. 2).

Now, the pyramid  $K-FfH$  may be regarded as having  $FKH$  for its base and  $f$  for its vertex. From  $K$ , draw  $KL$  parallel to  $GH$ ; it will be parallel to  $gh$ : then will the triangle  $FKL$  be equal to  $fgh$ , for the side  $FK$  is equal to  $fg$ , the angle  $F$  to the angle  $f$ , and the angle  $K$  to the angle  $g$ . But,  $FKH$  is a mean proportional between  $FKL$  and  $FGH$  (B. IV., P. XXIV., C.), or between  $fgh$  and  $FGH$ . The pyramid  $f-FKH$ , has, therefore, for its base a mean proportional between the upper and lower bases of the frustum, and its altitude is equal to that of the frustum; but the pyramid  $f-FKH$  is equal in volume to the pyramid  $g-FfH$ : hence, the volume of the given frustum is equal to that of three pyramids whose common altitude is equal to that of the frustum, and whose bases are the upper base, the lower base, and a mean proportional between them; which was to be proved.



*Cor. The volume of the frustum of any pyramid is equal to the sum of the volumes of three pyramids whose common altitude is that of the frustum, and whose bases are the lower base of the frustum, the upper base of the frustum, and a mean proportional between them.*

For, let  $ABCDE-e$  be a frustum of any pyramid. Through any lateral edge, as  $eE$ , pass the planes  $eEBb$ ,  $eECc$ , dividing it into triangular frustums. Now, the sum of the volumes of the triangular frustums is equal to the sum of three sets of pyramids, whose common altitude is that of the given frustum. The bases of the first set make up the lower base of the given frustum, the bases of the second set make up the upper base of the given frustum, and the bases of the third set make up a mean proportional between the upper and lower base of the given frustum: hence, the sum of the volumes of the first set is equal to that of a pyramid whose altitude is that of the frustum, and whose base is the lower base of the frustum; the sum of the volumes of the second set is equal to that of a pyramid whose altitude is that of the frustum, and whose base is the upper base of the frustum; and, the sum of the third set is equal to that of a pyramid whose altitude is that of the frustum, and whose base is a mean proportional between the two bases.



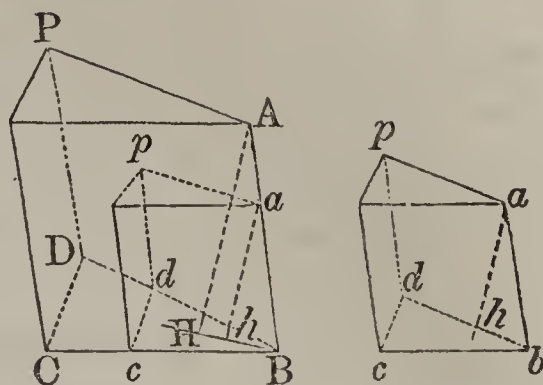
#### PROPOSITION XIX. THEOREM.

*Similar triangular prisms are to each other as the cubes of their homologous edges.*

Let  $CBD-P$ ,  $cbd-p$ , be two similar triangular prisms, and let  $BC$ ,  $bc$ , be any two homologous edges: then will the prism  $CBD-P$  be to the prism  $cbd-p$ , as  $\overline{BC}^3$  to  $\overline{bc}^3$ ,



For, the homologous angles  $B$  and  $b$  are equal, and the faces which bound them are similar (D. 16): hence, these trihedral angles may be applied, one to the other, so that the angle  $cbd$  will coincide with  $CBD$ , the edge  $ba$  with  $BA$ . In this case, the prism  $cbd-p$  will take the position  $Bcd-p$ . From  $A$  draw  $AH$  perpendicular to



the common base of the prisms: then will the plane  $BAH$  be perpendicular to the plane of the common base (B. VI, P. XVI.). From  $a$ , in the plane  $BAH$ , draw  $ah$  perpendicular to  $BH$ : then will  $ah$  also be perpendicular to the base  $BDC$  (B. VI, P. XVII.); and  $AH$ ,  $ah$ , will be the altitudes of the two prisms.

Since the bases  $CBD$ ,  $cbd$ , are similar, we have (B. IV., P. XXV.),

$$\text{base } CBD : \text{base } cbd :: \overline{CB}^2 : \overline{cb}^2.$$

Now, because of the similar triangles  $ABH$ ,  $aBh$ , and of the similar parallelograms  $AC$ ,  $ac$ , we have,

$$AH : ah :: CB : cb;$$

hence, multiplying these proportions term by term, we have,

$$\text{base } CBD \times AH : \text{base } cbd \times ah :: \overline{CB}^3 : \overline{cb}^3.$$

But,  $\text{base } CBD \times AH$  is equal to the volume of the prism  $CDB-A$ , and  $\text{base } cbd \times ah$  is equal to the volume of the prism  $cbd-p$ ; hence,

$$\text{prism } CDB-P : \text{prism } cbd-p :: \overline{CB}^3 : \overline{cb}^3;$$

which was to be proved.

*Cor. 1. Any two similar prisms are to each other as the cubes of their homologous edges.*

For, since the prisms are similar, their bases are similar polygons (D. 16); and these similar polygons may each be divided into the same number of similar triangles, similarly placed (B. IV., P. XXVI.); therefore, each prism may be divided into the same number of triangular prisms, having their faces similar and like placed; consequently, the triangular prisms are similar (D. 16). But, these triangular prisms are to each other as the cubes of their homologous edges, and being like parts of the polygonal prisms, the polygonal prisms themselves are to each other as the cubes of their homologous edges.

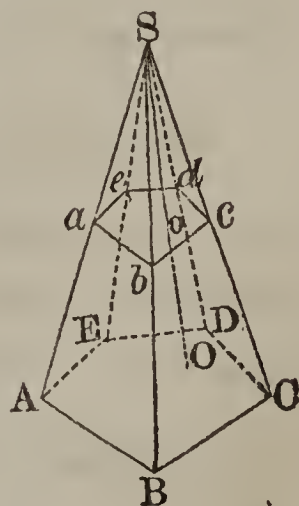
*Cor. 2. Similar prisms are to each other as the cubes of their altitudes, or as the cubes of any other homologous lines.*

#### PROPOSITION XX. THEOREM.

*Similar pyramids are to each other as the cubes of their homologous edges.*

Let  $S-ABCDE$ , and  $S-abcde$ , be two similar pyramids, so placed that their homologous angles at the vertex shall coincide, and let  $AB$  and  $ab$  be any two homologous edges: then will the pyramids be to each other as the cubes of  $AB$  and  $ab$ .

For, the face  $SAB$ , being similar to  $Sab$ , the edge  $AB$  is parallel to the edge  $ab$ , and the face  $SBC$  being similar to  $Sbc$ , the edge  $BC$  is parallel to  $bc$ ; hence, the planes of the bases are parallel (B. VI., P. XIII.).

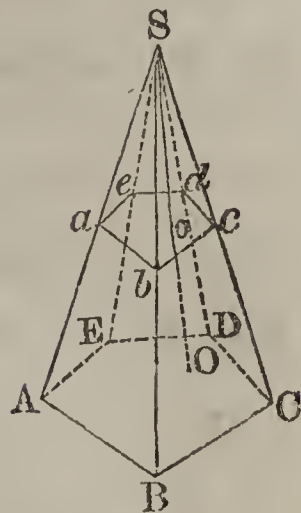


Draw  $SO$  perpendicular to the base  $ABCDE$ ; it will also be perpendicular to the base  $abcde$ . Let it pierce that plane at the point  $o$ : then will  $SO$  be to  $So$ , as  $SA$  is to  $Sa$  (P. III.), or as  $AB$  is to  $ab$ ; hence,

$$\frac{1}{3}SO : \frac{1}{3}So :: AB : ab.$$

But the bases being similar polygons, we have (B. IV., P. XXVII.),

$$\text{base } ABCDE : \text{base } abcde :: \overline{AB}^2 : \overline{ab}^2.$$



Multiplying these proportions, term by term, we have,

$$\text{base } ABCDE \times \frac{1}{3}SO : \text{base } abcde \times \frac{1}{3}So :: \overline{AB}^3 : \overline{ab}^3.$$

But,  $\text{base } ABCDE \times \frac{1}{3}SO$  is equal to the volume of the pyramid  $S-ABCDE$ , and  $\text{base } abcde \times \frac{1}{3}So$  is equal to the volume of the pyramid  $S-abcde$ ; hence,

$$\text{pyramid } S-ABCDE : \text{pyramid } S-abcde :: \overline{AB}^3 : \overline{ab}^3;$$

which was to be proved.

*Cor.* Similar pyramids are to each other as the cubes of their altitudes, or as the cubes of any other homologous lines.



## GENERAL FORMULAS.

If we denote the volume of any prism by  $V$ , its base by  $B$ , and its altitude by  $H$ , we shall have (P. XIV.),

$$V = B \times H \cdot \cdot \cdot \cdot \cdot (1.)$$

If we denote the volume of any pyramid by  $V$ , its base by  $B$ , and its altitude by  $H$ , we have (P. XVII.),

$$V = \frac{1}{3} B \times H \cdot \cdot \cdot \cdot \cdot (2.)$$

If we denote the volume of the frustum of any pyramid by  $V$ , its lower base by  $B$ , its upper base by  $b$ , and its altitude by  $H$ , we shall have (P. XVIII., C.),

$$V = \frac{1}{3} (B + b + \sqrt{B \times b}) \times H \cdot \cdot (3.)$$

## REGULAR POLYEDRONS.

A REGULAR POLYEDRON is one whose faces are all equal regular polygons.

There are five regular polyedrons, namely :

1. The TETRAEDRON, or *regular pyramid*—a polyedron bounded by four equal equilateral triangles.

2. The HEXAEDRON, or *cube*—a polyedron bounded by six equal squares.

3. The OCTAEDRON—a polyedron bounded by eight equal equilateral triangles.

4. The DODECAEDRON—a polyedron bounded by twelve equal and regular pentagons.

5. The ICOSAEDRON—a polyedron bounded by twenty equal equilateral triangles.

In the Tetraedron, the triangles are grouped about the polyedral angles in sets of three, in the Octaedron they are grouped in sets of four, and in the Icosaedron they are grouped in sets of five. Now, a greater number of equilateral triangles cannot be grouped so as to form a salient polyedral angle; for, if they could, the sum of the plane angles formed by the edges would be equal to, or greater than, four right angles, which is impossible (B. VI., P. XX.).

In the Hexaedron, the squares are grouped about the polyedral angles in sets of three. Now, a greater number of squares cannot be grouped so as to form a salient polyedral angle; for the same reason as before.

In the Dodecaedron, the regular pentagons are grouped about the polyedral angles in sets of three, and for the same reason as before, they cannot be grouped in any greater number, so as to form a salient polyedral angle.

Furthermore, no other regular polygons can be grouped so as to form a salient polyedral angle; therefore,

*Only five* regular polyedrons can be formed.

## BOOK VIII.

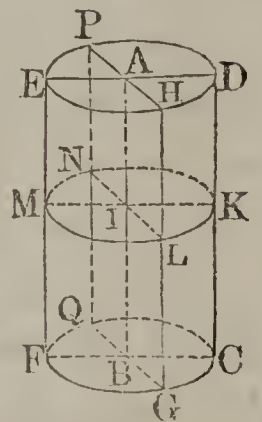
### THE CYLINDER, THE CONE, AND THE SPHERE.

#### DEFINITIONS.

1. A **CYLINDER** is a volume which may be generated by a rectangle revolving about one of its sides as an *axis*.

Thus, if the rectangle  $ABCD$  be turned about the side  $AB$ , as an axis, it will generate the cylinder  $FGCQ-P$ .

The fixed line  $AB$  is called *the axis of the cylinder*; the curved surface generated by the side  $CD$ , opposite the axis, is called *the convex surface of the cylinder*; the equal circles  $FGCQ$ , and  $EHDP$ , generated by the remaining sides  $BC$  and  $AD$ , are called *bases of the cylinder*; and the perpendicular distance between the planes of the bases, is called *the altitude of the cylinder*.



The line  $DC$ , which generates the convex surface, is, in any position, called an *element of the surface*; the elements are all perpendicular to the planes of the bases, and any one of them is equal to the altitude of the cylinder.

Any line of the generating rectangle  $ABCD$ , as  $IK$ , which is perpendicular to the axis, will generate a circle whose plane is perpendicular to the axis, and which is equal to either base: hence, any section of a cylinder by a plane perpendicular to the axis, is a circle equal to either base. Any section,  $FCDE$ , made by a plane through the axis, is a rectangle double the generating rectangle.

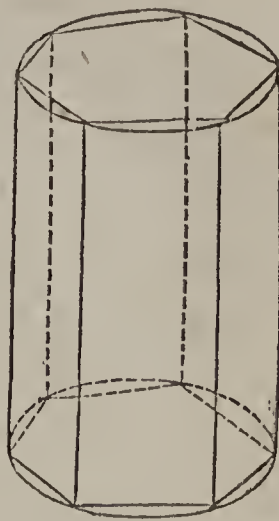


2. SIMILAR CYLINDERS are those which may be generated by similar rectangles revolving about homologous sides.

The axes of similar cylinders are proportional to the radii of their bases (B. IV., D. 1); they are also proportional to any other homologous lines of the cylinders.

3. A prism is said to be *inscribed in a cylinder*, when its bases are inscribed in the bases of the cylinder. In this case, the cylinder is said to be circumscribed about the prism.

The lateral edges of the inscribed prism are elements of the surface of the circumscribing cylinder.



4. A prism is said to be *circumscribed about a cylinder*, when its bases are circumscribed about the bases of the cylinder. In this case, the cylinder is said to be *inscribed in the prism*.

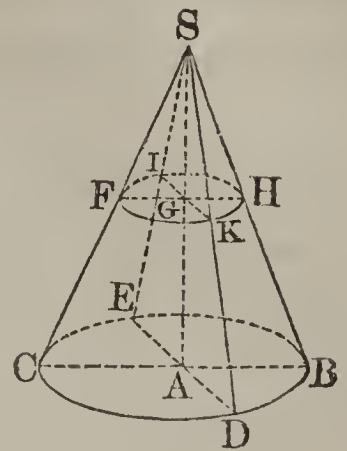
The lines which join the corresponding points of contact in the upper and lower bases, are common to the surface of the cylinder and to the lateral faces of the prism, and they are the only lines which are common. The lateral faces of the prism are said to be *tangent* to the cylinder along these lines, which are then called *elements of contact*.



5. A CONE is a volume which may be generated by a right-angled triangle revolving about one of the sides adjacent to the right angle, as an axis.

Thus, if the triangle  $SAB$ , right-angled at  $A$ , be turned about the side  $SA$ , as an axis, it will generate the cone  $S-CDBE$ .

The fixed line  $SA$ , is called *the axis of the cone*; the curved surface generated by the hypotenuse  $SB$ , is called *the convex surface of the cone*; the circle generated by the side  $AB$ , is called *the base of the cone*; and the point  $S$ , is called *the vertex of the cone*; the distance from the vertex to any point in the circumference of the base, is called *the slant height of the cone*; and the perpendicular distance from the vertex to the plane of the base, is called *the altitude of the cone*.



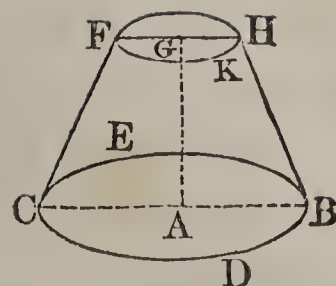
The line  $SB$ , which generates the convex surface, is, in any position, called an *element of the surface*; the elements are all equal, and any one is equal to the slant height; the axis is equal to the altitude.

Any line of the generating triangle  $SAB$ , as  $GH$ , which is perpendicular to the axis, generates a circle whose plane is perpendicular to the axis: hence, any section of a cone by a plane perpendicular to the axis, is a circle. Any section  $SBC$ , made by a plane through the axis, is an isosceles triangle, double the generating triangle.

6. A TRUNCATED CONE is that portion of a cone included between the base and any plane which cuts the cone.

When the cutting plane is parallel to the plane of the base, the truncated cone is called a FRUSTUM OF A CONE, and the intersection of the cutting plane with the cone is called the *upper base* of the frustum; the base of the cone is called the *lower base* of the frustum.

If the trapezoid  $HGAB$ , right-angled  $A$  and  $G$ , be revolved about  $AG$ , as an axis, it will generate a frustum of a cone, whose bases are  $ECDB$  and  $FKH$ , whose altitude is  $AG$ , and whose slant height is  $BH$ .

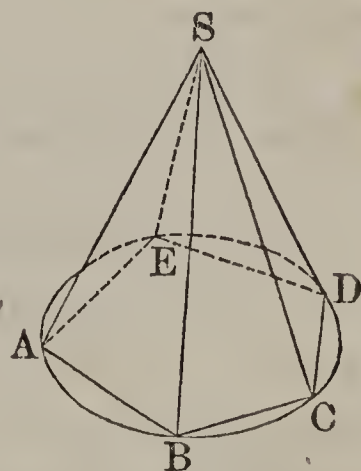


7. **SIMILAR CONES** are those which may be generated by similar right-angled triangles revolving about homologous sides.

The axes of similar cones are proportional to the radii of their bases (B. IV., D. 1); they are also proportional to any other homologous lines of the cones.

8. A pyramid is said to be *inscribed in a cone*, when its base is inscribed in the base of the cone, and when its vertex coincides with that of the cone.

The lateral edges of the inscribed pyramid are elements of the surface of the circumscribing cone.



9. A pyramid is said to be *circumscribed about a cone*, when its base is circumscribed about the base of the cone, and when its vertex coincides with that of the cone.

In this case, the cone is said to be *inscribed in the pyramid*.

The lateral faces of the circumscribing pyramid are tangent to the surface of the inscribed cone, along lines which are called *elements of contact*.

10. A frustum of a pyramid is *inscribed in a frustum*



of a cone, when its bases are inscribed in the bases of the frustum of the cone.

The lateral edges of the inscribed frustum of a pyramid are elements of the surface of the circumscribing frustum of a cone.

11. A frustum of a pyramid is circumscribed about a frustum of a cone, when its bases are circumscribed about those of the frustum of the cone.

Its lateral faces are tangent to the surface of the frustum of the cone, along lines which are called *elements of contact*.

12. A SPHERE is a volume bounded by a surface, every point of which is equally distant from a point within called the *centre*.

A sphere may be generated by a semicircle revolving about its diameter as an axis.

13. A RADIUS of a sphere is a straight line drawn from the centre to any point of the surface. A DIAMETER is any straight line drawn through the centre and limited at both extremities by the surface.

All the radii of a sphere are equal: the diameters are also equal, and each is double the radius.

14. A SPHERICAL SECTOR is a volume which may be generated by a sector of a circle revolving about a diameter of the circle lying without it.

The surface generated by the arc is called the *base of the sector*.

15. A plane is TANGENT TO A SPHERE when it touches it in a single point.

16. A ZONE is a portion of the surface of a sphere included between two parallel planes. The bounding lines

of the sections are called *bases* of the zone, and the distance between the planes is called the *altitude* of the zone.

If one of the planes is tangent to the sphere, the zone has but one base.

17. A SPHERICAL SEGMENT is a portion of a sphere included between two parallel planes. The sections made by the planes are called *bases* of the segment, and the distance between them is called the *altitude of the segment*.

If one of the planes is tangent to the sphere, the segment has but one base.

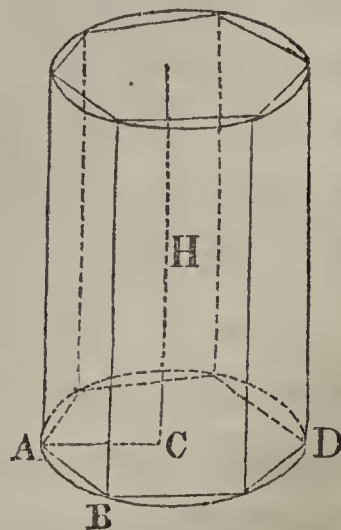
The CYLINDER, the CONE, and the SPHERE, are sometimes called THE THREE ROUND BODIES.

#### PROPOSITION I. THEOREM.

*The convex surface of a cylinder is equal to the circumference of its base multiplied by the altitude.*

Let  $ABD$  be the base of a cylinder whose altitude is  $H$ : then will its convex surface be equal to the circumference of its base multiplied by the altitude.

For, inscribe within the cylinder a prism whose base is a regular polygon. The convex surface of this prism will be equal to the perimeter of its base multiplied by its altitude (B. VII., P. I.), whatever may be the number of sides of its base. But, when the number of sides is infinite (B. V., P. X., C. 1), the convex surface of the prism coincides with that of the cylinder, the perimeter of



the base of the prism coincides with the circumference of the base of the cylinder, and the altitude of the prism is the same as that of the cylinder: hence, the convex surface of the cylinder is equal to the circumference of its base multiplied by the altitude; *which was to be proved.*

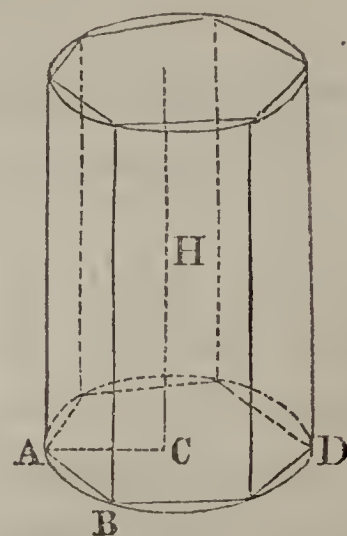
*Cor.* The convex surfaces of cylinders having equal altitudes are to each other as the circumferences of their bases.

### PROPOSITION II. THEOREM.

*The volume of a cylinder is equal to the product of its base and altitude.*

Let  $ABD$  be the base of a cylinder whose altitude is  $H$ ; then will its volume be equal to the product of its base and altitude.

For, inscribe within it a prism whose base is a regular polygon. The volume of this prism is equal to the product of its base and altitude (B. VII., P. XIV.), whatever may be the number of sides of its base. But, when the number of sides is infinite, the prism coincides with the cylinder, the base of the prism with the base of the cylinder, and the altitude of the prism is the same as that of the cylinder: hence, the volume of the cylinder is equal to the product of its base and altitude; *which was to be proved.*



*Cor. 1.* Cylinders are to each other as the products of their bases and altitudes; cylinders having equal bases are to each other as their altitudes; cylinders having equal altitudes are to each other as their bases.



*Cor. 2.* Similar cylinders are to each other as the cubes of their altitudes, or as the cubes of the radii of their bases.

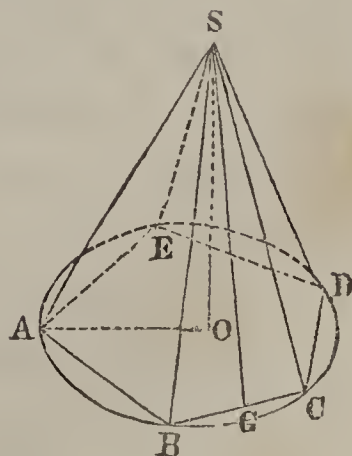
For, the bases are as the squares of their radii (B. V., P. XIII.), and the cylinders being similar, these radii are to each other as their altitudes (D. 2): hence, the bases are as the squares of the altitudes; therefore, the bases multiplied by the altitudes, or the cylinders themselves, are as the cubes of the altitudes.

PROPOSITION III. THEOREM.

*The convex surface of a cone is equal to the circumference of its base multiplied by half the slant height.*

Let  $S-ACD$  be a cone whose base is  $ACD$ , and whose slant height is  $SA$ : then will its convex surface be equal to the circumference of its base multiplied by half the slant height.

For, inscribe within it a right pyramid. The convex surface of this pyramid is equal to the perimeter of its base multiplied by half the slant height (B. VII., P. IV.), whatever may be the number of sides of its base. But when the number of sides of the base is infinite, the convex surface coincides with that of the cone, the perimeter of the base of the pyramid coincides with the circumference of the base of the cone, and the slant height of the pyramid is equal to the slant height of the cone: hence, the convex surface of the cone is equal to the circumference of its base multiplied by half the slant height; *which was to be proved.*

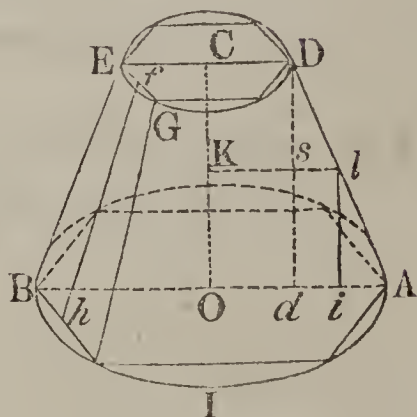


## PROPOSITION IV. THEOREM.

*The convex surface of a frustum of a cone is equal to half the sum of the circumferences of its two bases multiplied by the slant height.*

Let  $BIA-D$  be a frustum of a cone,  $BIA$  and  $EGD$  its two bases, and  $EB$  its slant height: then is its convex surface equal to half the sum of the circumferences of its two bases multiplied by its slant height.

For, inscribe within it the frustum of a right pyramid. The convex surface of this frustum is equal to half the sum of the perimeters of its bases, multiplied by the slant height (B. VII., P. IV., C.), whatever may be the number of its lateral faces. But when the number of these faces is infinite, the convex surface of the frustum of the pyramid coincides with that of the cone, the perimeters of its bases coincide with the circumferences of the bases of the frustum of the cone, and its slant height is equal to that of the cone: hence, the convex surface of the frustum of a cone is equal to half the sum of the circumferences of its bases multiplied by the slant height; *which was to be proved.*



*Scholium.* From the extremities  $A$  and  $D$ , and from the middle point  $l$ , of a line  $AD$ , let the lines  $AO$ ,  $DC$ , and  $lK$ , be drawn perpendicular to a line  $OC$ : then will  $lK$  be equal to half the sum of  $AO$  and  $DC$ . For, draw  $Dd$  and  $li$ , perpendicular to  $AO$ : then, because  $Al$  is equal to  $lD$ , we shall have  $Ai$  equal to  $id$  (B. IV., P. XV.), and consequently to  $ls$ ; that is,  $AO$  exceeds  $lK$

as much as  $LK$  exceeds  $DC$ : hence,  $LK$  is equal to the half sum of  $AO$  and  $DC$ .

Now, if the line  $AD$  be revolved about  $OC$ , as an axis, it will generate the surface of a frustum of a cone whose slant height is  $AD$ ; the point  $l$  will generate a circumference which is equal to half the sum of the circumferences generated by  $A$  and  $D$ : hence, *if a straight line be revolved about another straight line, it will generate a surface whose measure is equal to the product of the generating line and the circumference generated by its middle point.*

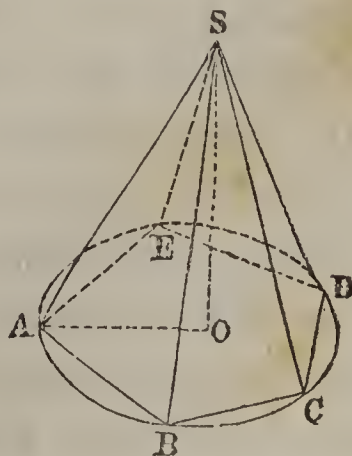
This proposition holds true when the line  $AD$  meets  $OC$ , and also when  $AD$  is parallel to  $OC$ .

#### PROPOSITION V. THEOREM.

*The volume of a cone is equal to its base multiplied by one-third of its altitude.*

Let  $ABDE$  be the base of a cone whose vertex is  $S$ , and whose altitude is  $So$ : then will its volume be equal to the base multiplied by one-third of the altitude.

For, inscribe in the cone a right pyramid. The volume of this pyramid is equal to its base multiplied by one-third of its altitude (B. VII., P. XVII.), whatever may be the number of its lateral faces. But, when the number of lateral faces is infinite, the pyramid coincides with the cone, the base of the pyramid coincides with that of the cone, and their altitudes are equal: hence, the volume of a cone is equal to the base multiplied by one-third of the altitude; *which was to be proved.*







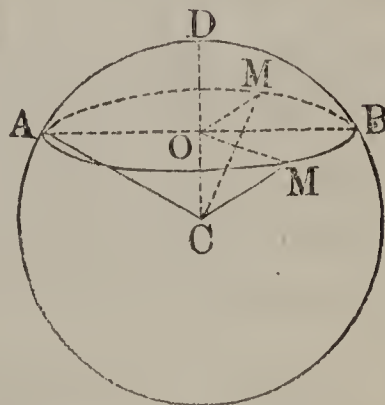
are equal to that of the frustum; hence, the volume of the frustum of a cone is equal to the sum of the volumes of three cones whose common altitude is that of the frustum, and whose bases are the lower base of the frustum, the upper base of the frustum, and a mean proportional between them; *which was to be proved.*

## PROPOSITION VII. THEOREM.

*Any section of a sphere made by a plane, is a circle.*

Let  $C$  be the centre of a sphere,  $CA$  one of its radii, and  $AMB$  any section made by a plane: then will this section be a circle.

For, draw a radius  $CO$  perpendicular to the cutting plane, and let it pierce the plane of the section at  $O$ . Draw radii of the sphere to any two points  $M, M'$ , of the curve which bounds the section, and join these points with  $O$ : then, because the radii  $CM, CM'$  are equal, the points  $M, M'$ , will be equally distant from  $O$  (B. VI., P. V., C.); hence, the section is a circle; *which was to be proved.*



*Cor. 1.* When the cutting plane passes through the centre of the sphere, the radius of the section is equal to that of the sphere; when the cutting plane does not pass through the centre of the sphere, the radius of the section will be less than that of the sphere.

A section whose plane passes through the centre of the sphere, is called a *great circle* of the sphere. A section whose plane does not pass through the centre of the sphere,

is called a *small circle* of the sphere. All great circles of the same, or of equal spheres, are equal.

*Cor. 2.* Any great circle divides the sphere, and also the surface of the sphere, into equal parts. For, the parts may be so placed as to coincide, otherwise there would be some points of the surface unequally distant from the centre, which is impossible.

*Cor. 3.* The centre of a sphere, and the centre of any small circle of that sphere, are in a straight line perpendicular to the plane of the circle.

*Cor. 4.* The square of the radius of any small circle is equal to the square of the radius of the sphere diminished by the square of the distance from the centre of the sphere to the plane of the circle (B. IV., P. XI., C. 1): hence, circles which are equally distant from the centre, are equal; and of two circles which are unequally distant from the centre, that one is the less whose plane is at the greater distance from the centre.

*Cor. 5.* The circumference of a great circle may always be made to pass through any two points on the surface of a sphere. For, a plane can always be passed through these points and the centre of the sphere (B. VI., P. II.), and its section will be a great circle. If the two points are the extremities of a diameter, an infinite number of planes can be passed through them and the centre of the sphere (B. VI., P. I., S.); in this case, an infinite number of great circles can be made to pass through the two points.

*Cor. 6.* The bases of a zone are the circumferences of circles (D. 16), and the bases of a segment of a sphere are circles.

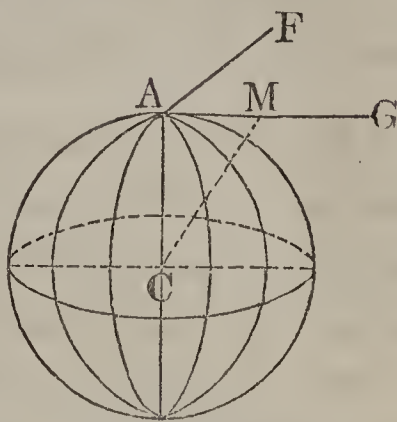


## PROPOSITION VIII. THEOREM.

*Any plane perpendicular to a radius of a sphere at its extremity, is tangent to the sphere at that point.*

Let  $C$  be the centre of a sphere,  $CA$  any radius, and  $FAG$  a plane perpendicular to  $CA$  at  $A$ : then will the plane  $FAG$  be tangent to the sphere at  $A$ .

For, from any other point of the plane, as  $M$ , draw the line  $MC$ : then because  $CA$  is a perpendicular to the plane, and  $CM$  an oblique line,  $CM$  will be greater than  $CA$  (B. VI., P. V.): hence, the point  $M$  lies without the sphere. The plane  $FAG$ , therefore, touches the sphere at  $A$ , and consequently is tangent to it at that point; *which was to be proved.*



*Scholium.* It may be shown, by a course of reasoning analogous to that employed in Book III., Propositions XI., XII., XIII., and XIV., that two spheres may have any one of six positions with respect to each other, viz.:

1°. When the distance between their centres is greater than the sum of their radii, *they are external, one to the other*:

2°. When the distance is equal to the sum of their radii, *they are tangent, externally*:

3°. When this distance is less than the sum, and greater than the difference of their radii, *they intersect each other*:

4°. When this distance is equal to the difference of their radii, *they are tangent internally*:

5°. When this distance is less than the difference of their radii, *one is wholly within the other*:

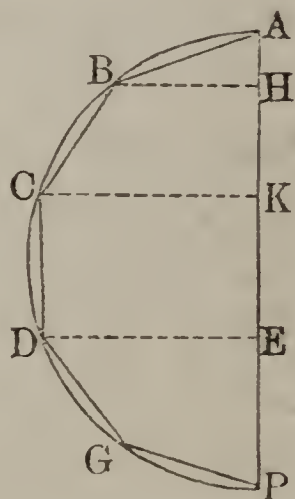
6°. When this distance is equal to zero, *they have a common centre, or, are concentric.*

## DEFINITIONS.

1°. If a semi-circumference be divided into equal arcs, the chords of these arcs form half of the perimeter of a regular inscribed polygon; this half perimeter is called a *regular semi-perimeter*. The figure bounded by the regular semi-perimeter and the diameter of the semi-circumference is called a *regular semi-polygon*. The diameter itself is called the *axis* of the semi-polygon.

2°. If lines be drawn from the extremities of any side, and perpendicular to the axis, the intercepted portion of the axis is called the *projection* of that side.

The broken line  $ABCDGP$  is a regular semi-perimeter; the figure bounded by it and the diameter  $AP$ , is a regular semi-polygon,  $AP$  is its axis,  $HK$  is the projection of the side  $BC$ , and the axis,  $AP$ , is the projection of the entire semi-perimeter.



## PROPOSITION IX. LEMMA.

*If a regular semi-polygon be revolved about its axis, the surface generated by the semi-perimeter will be equal to the axis multiplied by the circumference of the inscribed circle.*

Let  $ABCDEF$  be a regular semi-polygon,  $AF$  its axis, and  $ON$  its apothem: then will the surface generated by the regular semi-perimeter be equal to  $AF \times \text{circ. } ON$ .

From the extremities of any side, as  $DE$ , draw  $DI$  and  $EII$  perpendicular to  $AF$ ; draw also  $NM$  perpendicular to  $AF$ , and  $EK$  perpendicular to  $DI$ . Now, the surface generated by  $ED$  is equal to  $DE \times \text{circ. } NM$



(P. IV., S.). But, because the triangles  $EDK$  and  $ONM$  are similar (B. IV., P. XXI.), we have,

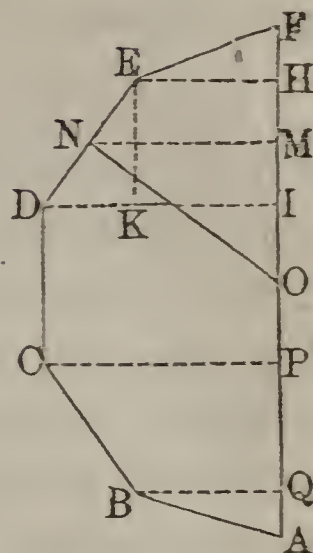
$$DE : EK \text{ or } IH :: ON : NM :: \text{circ. } ON : \text{circ. } NM;$$

whence,

$$DE \times \text{circ. } NM = IH \times \text{circ. } ON;$$

that is, the surface generated by any side is equal to the projection of that side multiplied by the circumference of the inscribed circle: hence, the surface generated by the entire semi-perimeter is equal to the sum of the projections of its sides, or the axis, multiplied by the circumference of the inscribed circle; *which was to be proved.*

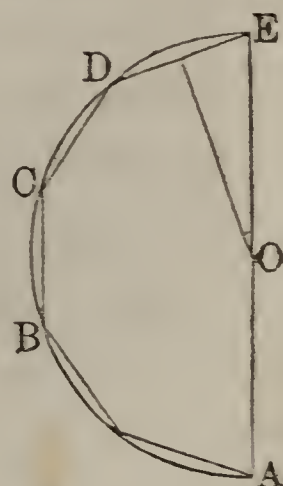
*Cor.* The surface generated by any portion of the perimeter, as  $CDE$ , is equal to its projection  $PH$ , multiplied by the circumference of the inscribed circle.



PROPOSITION X. THEOREM.

*The surface of a sphere is equal to its diameter multiplied by the circumference of a great circle.*

Let  $ABCDE$  be a semi-circumference,  $O$  its centre, and  $AE$  its diameter: then will the surface of the sphere generated by revolving the semi-circumference about  $AE$ , be equal to  $AE \times \text{circ. } OE$ .

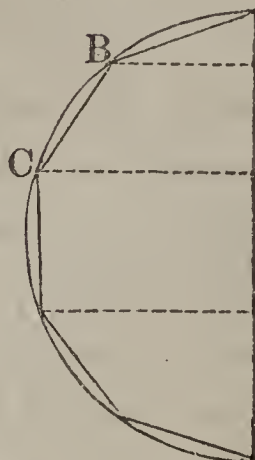


For, the semi-circumference may be regarded as a regular semi-perimeter with an infinite number of sides, whose axis is  $AE$ , and the radius of whose inscribed circle is  $OE$ : hence (P. IX.), the surface generated by it is equal to  $AE \times \text{circ. } OE$ ; *which was to be proved.*



*Cor. 1.* The circumference of a great circle is equal to  $2\pi OE$  (B. V., P. XVI.): hence, the area of the surface of the sphere is equal to  $2OE \times 2\pi OE$ , or to  $4\pi \overline{OE}^2$ ; that is, *the area of the surface of a sphere is equal to four great circles.*

*Cor. 2.* The surface generated by any arc of the semicircle, as  $BC$ , will be a zone, whose altitude is equal to the projection of that arc on the diameter. But, the arc  $BC$  is a portion of a semiperimeter having an infinite number of sides, and the radius of whose inscribed circle is equal to that of the sphere: hence (P. IX., C.), the surface of a zone is equal to its altitude multiplied by the circumference of a great circle of the sphere.



*Cor. 3.* Zones, on the same sphere, or on equal spheres, are to each other as their altitudes.

#### PROPOSITION XI. LEMMA.

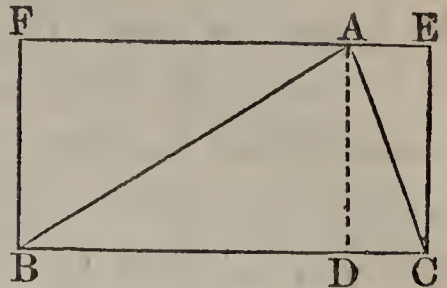
*If a triangle and a rectangle having the same base and equal altitudes, be revolved about the common base, the volume generated by the triangle will be one-third of that generated by the rectangle.*

Let  $ABC$  be a triangle, and  $EFBC$  a rectangle, having the same base  $BC$ , and an equal altitude  $AD$ , and let them both be revolved about  $BC$ : then will the volume generated by  $ABC$  be one-third of that generated by  $EFBC$ .

For, the cone generated by the right-angled triangle  $ADB$ , is equal to one-third of the cylinder generated by

the rectangle  $ADBF$  (P. V., C. 1); and the cone generated by the triangle  $ADC$ , is equal to one-third of the cylinder generated by the rectangle  $ADCE$ .

But, when  $AD$  falls within the triangle, the sum of the cones generated by  $ADB$  and  $ADC$ , is equal to the volume generated by the triangle  $ABC$ ; and the sum of the cylinders generated by  $ADBF$  and  $ADCE$ , is equal to the volume generated by the rectangle  $EFBC$ . When  $AD$  falls without the triangle, the difference of the cones generated by  $ADB$  and  $ADC$ , is equal to the volume generated by  $ABC$ ; and the difference of the cylinders generated by  $ADBF$  and  $ADCE$ , is equal to the volume generated by  $EFBC$ : hence, in either case, the volume generated by the triangle  $ABC$ , is equal to one-third of the volume generated by the rectangle  $EFBC$ ; which was to be proved.



*Cor.* The volume of the cylinder generated by  $EFBC$ , is equal to the product of its base and altitude, or to  $\pi \overline{AD}^2 \times BC$ : hence, the volume generated by the triangle  $ABC$ , is equal to  $\frac{1}{3} \pi \overline{AD}^2 \times BC$ .

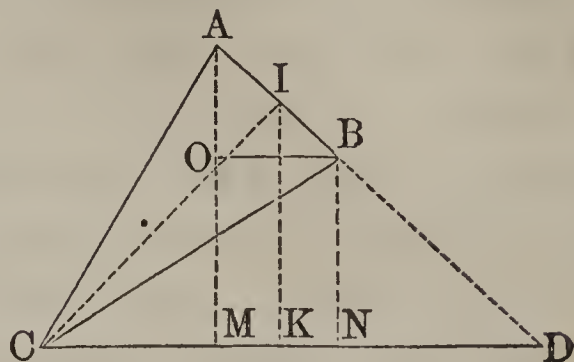
#### PROPOSITION XII. LEMMA.

*If an isosceles triangle be revolved about a straight line passing through its vertex, the volume generated will be equal to the surface generated by the base multiplied by one-third of the altitude.*

Let  $CAB$  be an isosceles triangle,  $C$  its vertex,  $AB$  its base,  $CI$  its altitude, and let it be revolved about the line  $CD$ , as an axis: then will the volume generated be equal to  $\text{surf. } AB \times \frac{1}{3} CI$ .

There may be two cases: the base, or base produced, may meet the axis; or, the base may be parallel to the axis.

1°. Suppose the base, when produced, to meet the axis at  $D$ ; draw  $AM$ ,  $IK$ , and  $BN$ , perpendicular to  $CD$ , and  $BO$  parallel to  $DC$ . Now, the volume generated by  $CAB$  is equal to the difference of the volumes generated by  $CAD$  and  $CBD$ ; hence (P. XI., C.),



$$\text{vol } CAB = \frac{1}{3}\pi \overline{AM}^2 \times CD - \frac{1}{3}\pi \overline{BN}^2 \times CD = \frac{1}{3}\pi (\overline{AM}^2 - \overline{BN}^2) \times CD.$$

But,  $\overline{AM}^2 - \overline{BN}^2$  is equal to  $(AM + BN)(AM - BN)$ , (B. IV., P. X.); and because  $AM + BN$  is equal to  $2IK$  (P. IV., S.), and  $AM - BN$  to  $AO$ , we have,

$$\text{vol. } CAB = \frac{2}{3}\pi IK \times AO \times CD.$$

But, the right-angled triangles  $AOB$  and  $CDI$  are similar (B. IV., P. XXI.); hence,

$$AO : AB :: CI : CD; \quad \text{or,} \quad AO \times CD = AB \times CI.$$

Substituting, and changing the order of the factors, we have,

$$\text{vol. } CAB = AB \times 2\pi IK \times \frac{1}{3}CI.$$

But,  $AB \times 2\pi IK$  is equal to the surface generated by  $AB$ ; hence,

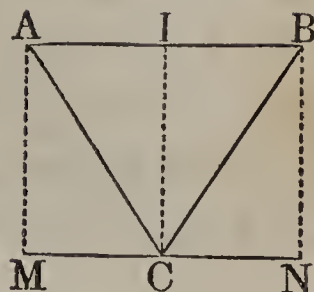
$$\text{vol. } CAB = \text{surf. } AB \times \frac{1}{3}CI.$$

This demonstration holds good when the axis  $CD$  coincides with one side of the triangle  $CAB$ .

2°. Suppose the base of the triangle to be parallel to the axis.



Draw  $AM$  and  $BN$  perpendicular to the axis. The volume generated by  $CAB$ , is equal to the cylinder generated by the rectangle  $ABNM$ , diminished by the sum of the cones generated by the triangles  $CAM$  and  $BCN$ ; hence,



$$\text{vol. } CAB = \pi \overline{CI}^2 \times AB - \frac{1}{3} \pi \overline{CI}^2 \times AI - \frac{1}{3} \pi \overline{CI}^2 \times IB.$$

But the sum of  $AI$  and  $IB$  is equal to  $AB$ : hence, we have, by reducing, and changing the order of the factors,

$$\text{vol. } CAB = AB \times 2 \pi CI \times \frac{1}{3} CI.$$

But  $AB \times 2 \pi CI$  is equal to the surface generated by  $AB$ ; consequently,

$$\text{vol. } CAB = \text{surf. } AB \times \frac{1}{3} CI;$$

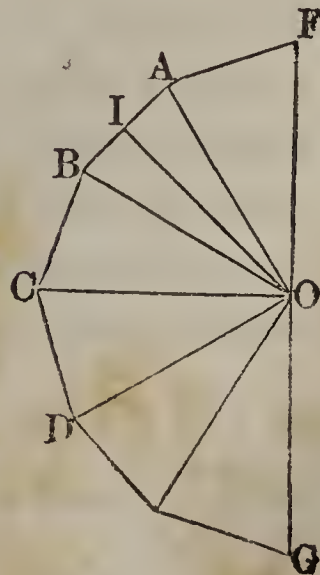
hence, in all cases, the volume generated by  $CAB$  is equal to  $\text{surf. } AB \times \frac{1}{3} CI$ ; which was to be proved.

### PROPOSITION XIII. LEMMA.

If a regular semi-polygon be revolved about its axis, the volume generated will be equal to the surface generated by the semi-perimeter multiplied by one-third of the apothem.

Let  $FBDG$  be a regular semi-polygon,  $FG$  its axis,  $OI$  its apothem, and let the semi-polygon be revolved about  $FG$ : then will the volume generated be equal to  $\text{surf. } FBDG \times \frac{1}{3} OI$ .

For, draw lines from the vertices to the centre  $O$ . These lines will divide the semi-polygon into isosceles triangles whose bases are sides of the semi-polygon, and whose altitudes are equal to  $OI$ .



Now, the sum of the volumes generated by these triangles is equal to the volume generated by the semi-polygon. But, the volume generated by any triangle, as  $OAB$ , is equal to  $\text{surf. } AB \times \frac{1}{3}OI$  (P. XII.) : hence, the volume generated by the semi-polygon is equal to  $\text{surf. } FB DG \times \frac{1}{3}OI$ ; *which was to be proved.*

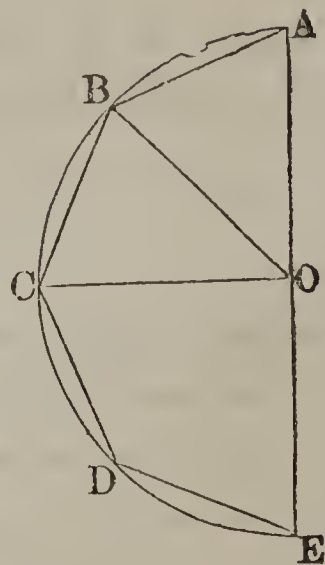
*Cor.* The volume generated by a portion of the semi-polygon,  $OABC$ , limited by radii  $OC$ ,  $OA$ , is equal to  $\text{surf. } ABC \times \frac{1}{3}OI$ .

PROPOSITION XIV. THEOREM.

*The volume of a sphere is equal to its surface multiplied by one-third of its radius.*

Let  $ACE$  be a semicircle,  $AE$  its diameter,  $O$  its centre, and let the semicircle be revolved about  $AE$ : then will the volume generated be equal to the surface generated by the semi-circumference multiplied by one-third of the radius  $OA$ .

For, the semicircle may be regarded as a regular semi-polygon having an infinite number of sides, whose semi-perimeter coincides with the semi-circumference, and whose apothem is equal to the radius: hence (P. XIII.), the volume generated by the semicircle is equal to the surface generated by the semi-circumference multiplied by one-third of the radius; *which was to be proved.*



*Cor. 1.* Any portion of the semicircle, as  $OBC$ , bounded by two radii, will generate a volume equal to the surface

generated by the arc  $BC$  multiplied by one-third of the radius (P. XIII., C.). But this portion of the semicircle is a circular sector, the volume which it generates is a spherical sector, and the surface generated by the arc is a zone: hence, the *volume of a spherical sector is equal to the zone which forms its base multiplied by one-third of the radius*

*Cor. 2.* If we denote the volume of a sphere by  $V$ , and its radius by  $R$ , the area of the surface will be equal to  $4\pi R^2$  (P. X., C. 1), and the volume of the sphere will be equal to  $4\pi R^2 \times \frac{1}{3}R$ ; consequently, we have,

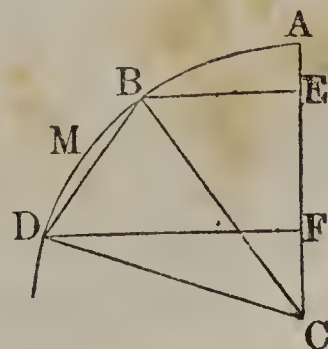
$$V = \frac{4}{3}\pi R^3.$$

Again, if we denote the diameter of the sphere by  $D$ , we shall have  $R$  equal to  $\frac{1}{2}D$ , and  $R^3$  equal to  $\frac{1}{8}D^3$ , and consequently,

$$V = \frac{1}{6}\pi D^3;$$

hence, *the volumes of spheres are to each other as the cubes of their radii, or as the cubes of their diameters.*

*Scholium.* If the figure  $EBDF$ , formed by drawing lines from the extremities of the arc  $BD$  perpendicular to  $CA$ , be revolved about  $CA$ , as an axis, it will generate a segment of a sphere whose volume may be found by adding to the spherical sector generated by  $CDB$ , the cone generated by  $CBE$ , and subtracting from their sum the cone generated by  $CDF$ . If the arc  $BD$  is so taken that the points  $E$  and  $F$  fall on opposite sides of the centre  $C$ , the latter cone must be added, instead of subtracted: hence,



$$\text{segment } EBDF = \text{zone } BD \times \frac{1}{3}CD + \pi \overline{BE}^2 \times \frac{1}{3}CE - \pi \overline{DF}^2 \times \frac{1}{3}CF.$$



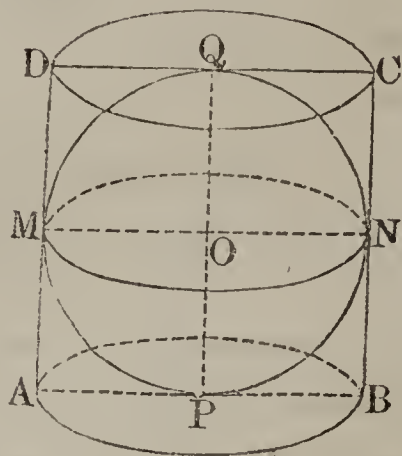
## PROPOSITION XV. THEOREM

*The surface of a sphere is to the entire surface of the circumscribed cylinder, including its bases, as 2 is to 3 : and the volumes are to each other in the same ratio.*

Let  $PMQ$  be a semicircle, and  $PADQ$  a rectangle, whose sides  $PA$  and  $QD$  are tangent to the semicircle at  $P$  and  $Q$ , and whose side  $AD$ , is tangent to the semicircle at  $M$ . If the semicircle and the rectangle be revolved about  $PQ$ , as an axis, the former will generate a sphere, and the latter a circumscribed cylinder.

1°. The surface of the sphere is to the entire surface of the cylinder, as 2 is to 3.

For, the surface of the sphere is equal to four great circles (P. X., C. 1), the convex surface of the cylinder is equal to the circumference of its base multiplied by its altitude (P. I.) ; that is, it is equal to the circumference of a great circle multiplied by its diameter, or to four great circles (B. V., P. XV.) ; adding to this the two bases, each of which is equal to a great circle, we have the entire surface of the cylinder equal to six great circles : hence, the surface of the sphere is to the entire surface of the circumscribed cylinder, as 4 is to 6, or as 2 is to 3 ; *which was to be proved.*



2°. The volume of the sphere is to the volume of the cylinder as 2 is to 3.

For, the volume of the sphere is equal to  $\frac{4}{3}\pi R^3$  (P. XIV., C. 2) ; the volume of the cylinder is equal to its base multiplied by its altitude (P. II.) ; that is, it is equal to

$\pi R^2 \times 2R$ , or to  $\frac{6}{3}\pi R^3$ : hence, the volume of the sphere is to that of the cylinder as 4 is to 6, or as 2 is to 3; which was to be proved.

*Cor.* The surface of a sphere is to the entire surface of a circumscribed cylinder, as the volume of the sphere is to volume of the cylinder.

*Scholium.* Any polyedron which is circumscribed about a sphere, that is, whose faces are all tangent to the sphere, may be regarded as made up of pyramids, whose bases are the faces of the polyedron, whose common vertex is at the centre of the sphere, and each of whose altitudes is equal to the radius of the sphere. But, the volume of any one of these pyramids is equal to its base multiplied by one-third of its altitude: hence, the volume of a circumscribed polyedron is equal to its surface multiplied by one-third of the radius of the inscribed sphere.

Now, because the volume of the sphere is also equal to its surface multiplied by one-third of its radius, it follows that the volume of a sphere is to the volume of any circumscribed polyedron, as the surface of the sphere is to the surface of the polyedron.

Polyedrons circumscribed about the same, or about equal spheres, are proportional to their surfaces.

#### GENERAL FORMULAS.

If we denote the convex surface of a cylinder by  $S$ , its volume by  $V$ , the radius of its base by  $R$ , and its altitude by  $H$ , we have (P. I., II.),

$$S = 2\pi R \times H \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.)$$

$$V = \pi R^2 \times H \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.)$$





## BOOK IX.

### SPHERICAL GEOMETRY.

#### DEFINITIONS.

1 A SPHERICAL ANGLE is an angle included between the arcs of two great circles of a sphere meeting at a point. The arcs are called *sides* of the angle, and the point at which they meet is called the *vertex* of the angle.

The measure of a spherical angle is the same as that of the diedral angle included between the planes of its sides. Spherical angles may be *acute*, *right*, or *obtuse*.

2. A SPHERICAL POLYGON is a portion of the surface of a sphere bounded by arcs of three or more great circles. The bounding arcs are called *sides* of the polygon, and the points in which the sides meet are called *vertices* of the polygon. Each side is supposed to be less than a semi-circumference.

Spherical polygons are classified in the same manner as plane polygons.

3. A SPHERICAL TRIANGLE is a spherical polygon of three sides.

Spherical triangles are classified in the same manner as plane triangles.

4. A LUNE is a portion of the surface of a sphere bounded by semi-circumferences of two great circles.

5. A SPHERICAL WEDGE is a portion of a sphere bounded by a lune and two semicircles meeting in a diameter of the sphere.

6. A SPHERICAL PYRAMID is a portion of a sphere bounded by a spherical polygon and sectors of circles whose common centre is the centre of the sphere.

The spherical polygon is called the *base* of the pyramid, and the centre of the sphere is called the *vertex* of the pyramid.

7. A POLE OF A CIRCLE is a point on the surface of the sphere, equally distant from all the points of the circumference of the circle.

8. A DIAGONAL of a spherical polygon is an arc of a great circle joining the vertices of any two angles which are not consecutive.

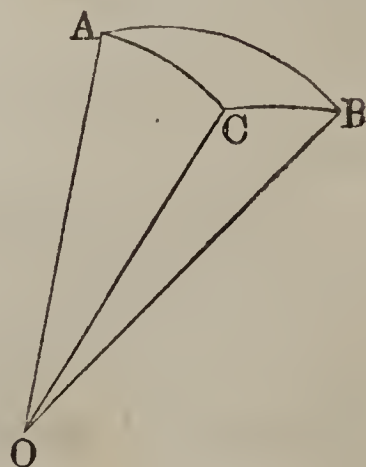
#### PROPOSITION I. THEOREM.

*Any side of a spherical triangle is less than the sum of the other two.*

Let  $ABC$  be a spherical triangle situated on a sphere whose centre is  $O$ : then will any side, as  $AB$ , be less than the sum of the sides  $AC$  and  $BC$ .

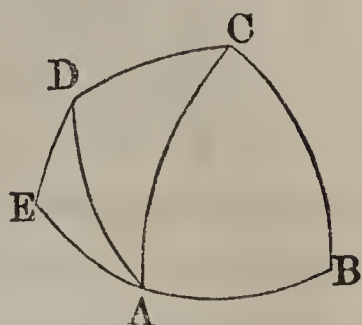
For, draw the radii  $OA$ ,  $OB$ , and  $OC$ : these radii form the edges of a trihedral angle whose vertex is  $O$ , and the plane angles included between them are measured by the arcs  $AB$ ,  $AC$ , and  $BC$  (B. III., P. XVII., Sch.). But any plane angle, as  $AOB$ , is less than the sum of the plane angles  $AOC$  and  $BOC$  (B. VI., P. XIX.): hence,

the arc  $AB$  is less than the sum of the arcs  $AC$  and  $BC$ ; which was to be proved.



*Cor. 1.* Any side  $AB$ , of a spherical polygon  $ABCDE$ , is less than the sum of all the other sides.

For, draw the diagonals  $AC$  and  $AD$ , dividing the polygon into triangles. The arc  $AB$  is less than the sum of  $AC$  and  $BC$ , the arc  $AC$  is less than the sum of  $AD$  and  $DC$ , and the arc  $AD$  is less than the sum of  $DE$  and  $EA$ ; hence,  $AB$  is less than the sum of  $BC$ ,  $CD$ ,  $DE$ , and  $EA$ .



*Cor. 2.* The arc of a great circle joining any two points on the surface of a sphere, is less than the arc of a small circle joining the same points.

For, divide the arc of the small circle into equal parts, and through the extremities of each part pass the arc of a great circle. The arc of the great circle joining the given points will be less than the sum of these arcs (C. 1), whatever may be their number. But when this number is infinite, the arcs of the great circle coincide with the corresponding arcs of the small circle, and their sum is equal to the entire arc of the small circle.

*Cor. 3.* The shortest distance between two points on the surface of a sphere, is measured on the arc of a great circle joining them.

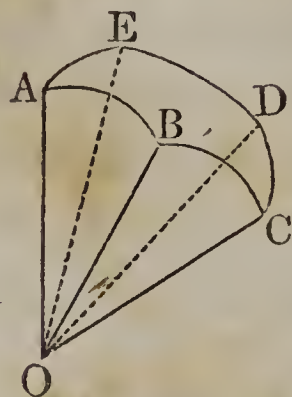
#### PROPOSITION II. THEOREM.

*The sum of the sides of a spherical polygon is less than the circumference of a great circle.*

Let  $ABCDE$  be a spherical polygon situated on a sphere whose centre is  $O$ : then will the sum of its sides be less than the circumference of a great circle.



For, draw the radii  $OA$ ,  $OB$ ,  $OC$ ,  $OD$ , and  $OE$ : these radii form the edges of a polyedral angle whose vertex is at  $O$ , and the angles included between them are measured by the arcs  $AB$ ,  $BC$ ,  $CD$ ,  $DE$ , and  $EA$ . But the sum of these angles is less than four right angles (B. VI., P. XX.): hence, the sum of the arcs which measure them is less than the circumference of a great circle; *which was to be proved.*

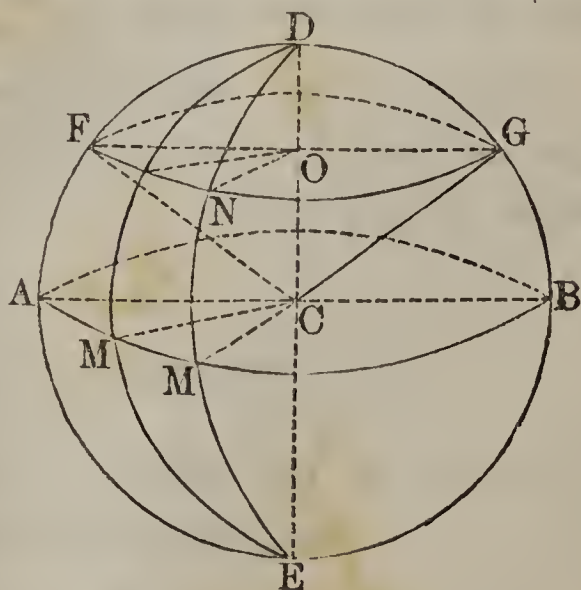


### PROPOSITION III. THEOREM.

*If a diameter of a sphere be drawn perpendicular to the plane of any circle of the sphere, its extremities will be poles of that circle.*

Let  $C$  be the centre of a sphere,  $FNG$  any circle of the sphere, and  $DE$  a diameter of the sphere perpendicular to the plane of  $FNG$ : then will the extremities  $D$  and  $E$ , be poles of the circle  $FNG$ .

The diameter  $DE$ , being perpendicular to the plane of  $FNG$ , must pass through the centre  $O$  (B. VIII., P. VII., C. 3). If arcs of great circles  $DN$ ,  $DF$ ,  $DG$ , &c., be drawn from  $D$  to different points of the circumference  $FNG$ , and chords of these arcs be drawn, these chords will be equal (B. VI.,



P. V.), consequently, the arcs themselves will be equal. But these arcs are the shortest lines that can be drawn from the

point  $D$ , to the different points of the circumference (P. I., C. 2): hence, the point  $D$ , is equally distant from all the points of the circumference, and consequently is a pole of the circle (D. 7). In like manner, it may be shown that the point  $E$  is also a pole of the circle: hence, both  $D$ , and  $E$ , are poles of the circle  $FNG$ ; which was to be proved.

*Cor. 1.* Let  $AMB$  be a great circle perpendicular to  $DE$ : then will the angles  $DCM$ ,  $ECM$ , &c., be right angles; and consequently, the arcs  $DM$ ,  $EM$ , &c., will each be equal to a quadrant (B. III., P. XVII., S.): hence, the two poles of a great circle are at equal distances from the circumference.

*Cor. 2.* The two poles of a small circle are at unequal distances from the circumference, the sum of the distances being equal to a semi-circumference.

*Cor. 3.* The line  $DC$  being perpendicular to the plane  $AMB$ , any plane, as  $DMC$ , passed through it, will also be perpendicular to the plane  $AMB$ : hence, the spherical angle  $DMA$ , is a right-angle; that is, if any point, in the circumference of a great circle, be joined with either pole by the arc of a great circle, such arc will be perpendicular to the circumference of the given circle.

*Cor. 4.* If the distance of a point  $D$ , from each of the points  $A$  and  $M$ , in the circumference of a great circle, is equal to a quadrant, the point  $D$ , is the pole of the arc  $AM$ .

For, let  $C$  be the centre of the sphere, and draw the radii  $CD$ ,  $CA$ ,  $CM$ . Since the angles  $ACD$ ,  $MCD$ , are right angles, the line  $CD$  is perpendicular to the two straight lines  $CA$ ,  $CM$ : it is, therefore, perpendicular to their



plane (B. VI., P. IV.): hence, the point  $D$ , is the pole of the arc  $AM$ .

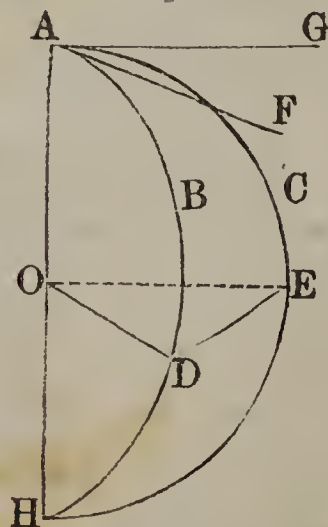
*Scholium.* The properties of these poles enable us to describe arcs of a circle on the surface of a sphere, with the same facility as on a plane surface. For, by turning the arc  $DF$  about the point  $D$ , the extremity  $F$  will describe the small circle  $FNG$ ; and by turning the quadrant  $DFA$  round the point  $D$ , its extremity  $A$  will describe an arc of a great circle.

#### PROPOSITION IV. THEOREM.

*The angle formed by two arcs of great circles, is equal to that formed by the tangents to these arcs at their point of intersection, and is measured by the arc of a great circle described from the vertex as a pole, and limited by the sides, produced if necessary.*

Let the angle  $BAC$  be formed by the two arcs  $AB$ ,  $AC$ : then is it equal to the angle  $FAG$  formed by the tangents  $AF$ ,  $AG$ , and is measured by the arc  $DE$  of a great circle, described about  $A$  as a pole.

For, the tangent  $AF$ , drawn in the plane of the arc  $AB$ , is perpendicular to the radius  $AO$ ; and the tangent  $AG$ , drawn in the plane of the arc  $AC$ , is perpendicular to the same radius  $AO$ : hence, the angle  $FAG$  is equal to the angle contained by the planes  $ABDH$ ,  $ACEH$  (B. VI., D. 4); which is that of the arcs  $AB$ ,  $AC$ . Now, if the arcs  $AD$  and  $AE$  are both quad-



rants, the lines  $OD$ ,  $OE$ , are perpendicular to  $OA$ , and

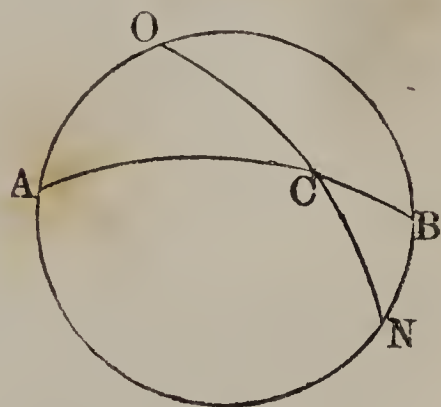


the angle  $DOE$  is equal to the angle of the planes  $ABDH$ ,  $ACEH$ : hence, the arc  $DE$  is the measure of the angle contained by these planes, or of the angle  $CAB$ ; which was to be proved.

*Cor. 1.* The angles of spherical triangles may be compared by means of the arcs of great circles described from their vertices as poles, and included between their sides.

A spherical angle can always be constructed equal to a given spherical angle.

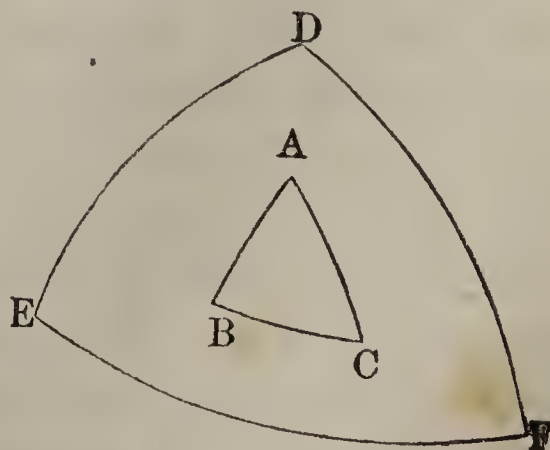
*Cor. 2.* Vertical angles, such as  $ACO$  and  $BCN$  are equal; for either of them is the angle formed by the two planes  $ACB$ ,  $OCN$ . When two arcs  $ACB$ ,  $OCN$ , intersect, the sum of two adjacent angles, as  $ACO$ ,  $OCB$ , is equal to two right angles.



#### PROPOSITION V. THEOREM.

*If from the vertices of the angles of a spherical triangle, as poles, arcs be described forming a spherical triangle, the vertices of the angles of this second triangle will be respectively poles of the sides of the first.*

From the vertices  $A$ ,  $B$ ,  $C$ , as poles, let the arcs  $EE'$ ,  $FD$ ,  $ED$ , be described, forming the triangle  $DFE$ : then will the vertices  $D$ ,  $E$ , and  $F$ , be respectively poles of the sides  $BC$ ,  $AC$ ,  $AB$ .



For, the point  $A$  being

the pole of the arc  $EF$ , the distance  $AE$ , is a quadrant; the point  $C$  being the pole of the arc  $DE$ , the distance  $CE$ , is likewise a quadrant: hence, the point  $E$  is at a quadrant's distance from the points  $A$  and  $C$ : hence, it is the pole of the arc  $AC$  (P. III., C. 4). It may be shown, in like manner, that  $D$  is the pole of the arc  $BC$ , and  $F$  that of the arc  $AB$ ; which was to be proved.

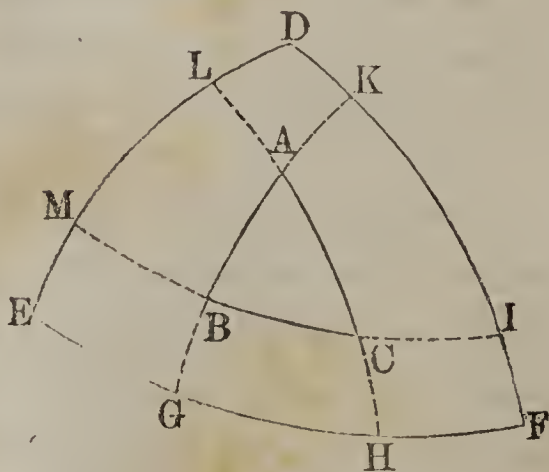
*Scholium.* The triangle  $ABC$ , may be described by means of  $DEF$ , as  $DEF$  is described by means of  $ABC$ . Triangles thus related are called *polar triangles*, or *supplemental triangles*.

#### PROPOSITION VI. THEOREM.

*Any angle, in one of two polar triangles, is measured by a semi-circumference, minus the side lying opposite to it in the other triangle.*

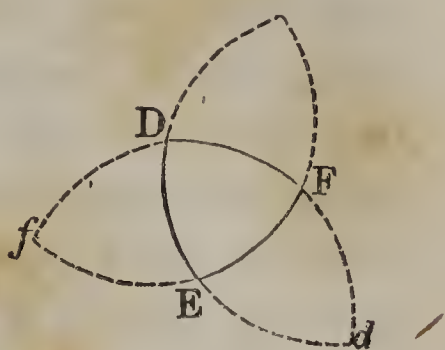
Let  $ABC$ , and  $EFD$ , be any two polar triangles: then will any angle in either triangle be measured by a semi-circumference, minus the side lying opposite to it in the other triangle.

For, produce the sides  $AB$ ,  $AC$ , if necessary, till they meet  $EF$ , in  $G$  and  $H$ . The point  $A$  being the pole of the arc  $GH$ , the angle  $A$  is measured by that arc (P. IV.). But, since  $E$  is the pole of  $AH$ , the arc  $EH$  is a quadrant; and since  $F$  is the pole of  $AG$ ,  $FG$  is a quadrant: hence, the sum of the arcs  $EH$  and  $GF$ , is equal to a semi-circumference. But,



the sum of the arcs  $EH$  and  $GF$ , is equal to the sum of the arcs  $EF$  and  $GH$ : hence, the arc  $GH$ , which measures the angle  $A$ , is equal to a semi-circumference, minus the arc  $EF$ . In like manner, it may be shown, that any other angle, in either triangle, is measured by a semi-circumference, minus the side lying opposite to it in the other triangle; *which was to be proved.*

*Scholium.* Besides the triangle  $DEF$ , three others may be formed by the intersection of the arcs  $DE$ ,  $EF$ ,  $DF$ . But the proposition is applicable only to the central triangle, which is distinguished from the other three by the circumstance, that the two vertices,  $A$  and  $D$ , lie on the same side of  $BC$ ; the two vertices,  $B$  and  $E$ , on the same side of  $AC$ ; and the two vertices,  $C$  and  $F$ , on the same side of  $AB$ .



#### PROPOSITION VII. THEOREM.

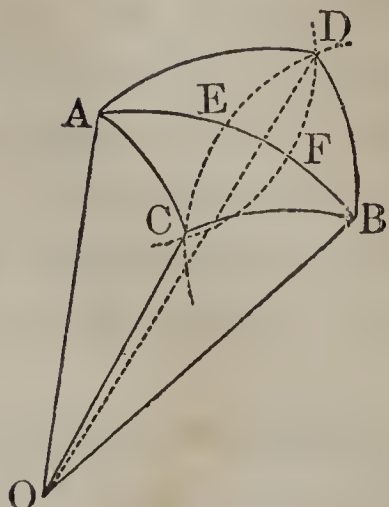
*If from the vertices of any two angles of a spherical triangle, as poles, arcs of circles be described passing through the vertex of the third angle; and if from the second point in which these arcs intersect, arcs of great circles be drawn to the vertices, used as poles, the parts of the triangle thus formed will be equal to those of the given triangle, each to each.*

Let  $ABC$  be a spherical triangle situated on a sphere whose centre is  $O$ ,  $CED$  and  $CFD$  arcs of circles described about  $B$  and  $A$  as poles, and let  $DA$  and  $DB$  be arcs of great circles: then will the parts of the



triangle  $ABD$  be equal to those of the given triangle  $ABC$ , each to each.

For, by construction, the side  $AD$  is equal to  $AC$ , the side  $DB$  is equal to  $BC$ , and the side  $AB$  is common: hence, the sides are equal, each to each. Draw the radii  $OA$ ,  $OB$ ,  $OC$ , and  $OD$ . The radii  $OA$ ,  $OB$ , and  $OC$ , will form the edges of a triedral angle whose vertex is  $O$ ; and the radii  $OA$ ,  $OB$ , and  $OD$ , will form the edges of a second triedral angle whose vertex is also at  $O$ ; and the plane angles formed by these edges will be equal, each to each: hence, the planes of the equal angles are equally inclined to each other (B. VI., P. XXI.). But, the angles made by these planes are equal to the corresponding spherical angles; consequently, the angle  $BAD$  is equal to  $BAC$ , the angle  $ABD$  to  $ABC$ , and the angle  $ADB$  to  $ACB$ : hence, the parts of the triangle  $ABD$  are equal to the parts of the triangle  $ACB$ , each to each; *which was to be proved.*



*Scholium 1.* The triangles  $ABC$  and  $ABD$ , are not, in general, capable of superposition, but their parts are *symmetrically* disposed with respect to  $AB$ . Triangles which can be so placed are called *symmetrical triangles*.

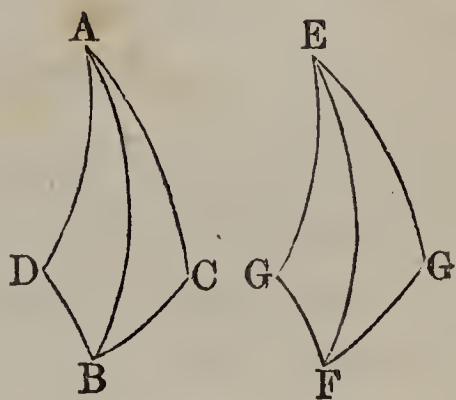
*Scholium 2.* If symmetrical triangles are isosceles, they can be so placed as to coincide throughout: hence, they are *equal in area*.

## PROPOSITION VIII. THEOREM.

*If two spherical triangles, on the same, or on equal spheres, have two sides and the included angle of the one equal to two sides and the included angle of the other, each to each, the remaining parts are equal, each to each.*

Let the spherical triangles  $ABC$  and  $EFG$ , have the side  $EF$  equal to  $AB$ , the side  $EG$  equal to  $AC$ , and the angle  $FEG$  equal to  $BAC$ : then will the side  $FG$  be equal to  $BC$ , the angle  $EFG$  to  $ABC$ , and the angle  $EGF$  to  $ACB$ .

For, the triangle  $EFG$  may be placed upon  $ABC$ , or upon its symmetrical triangle  $ADB$ , so as to coincide with it throughout, as may be shown by the same course of reasoning as that employed in Book I., Proposition V.: hence, the side  $FG$  is equal to  $BC$ , the angle  $EFG$  to  $ABC$ , and the angle  $EGF$  to  $ACB$ ; which was to be proved.



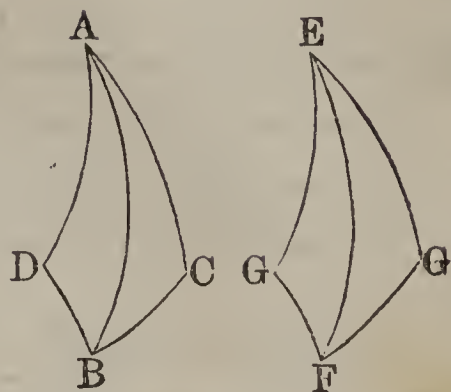
## PROPOSITION IX. THEOREM.

*If two spherical triangles on the same, or on equal spheres, have two angles and the included side of the one equal to two angles and the included side of the other, each to each, the remaining parts will be equal, each to each.*

Let the spherical triangles  $ABC$  and  $EFG$ , have the angle  $FEG$  equal to  $BAC$ , the angle  $EFG$  equal to  $ABC$ , and the side  $EF$  equal to  $AB$ : then will the

side  $EG$  be equal to  $AC$ , the side  $FG$  to  $BC$ , and the angle  $FGE$  to  $BCA$ .

For, the triangle  $EFG$  may be placed upon  $ABC$ , or upon its symmetrical triangle  $ADB$ , so as to coincide with it throughout, as may be shown by the same course of reasoning as that employed in Book I., Proposition VI.: hence, the side  $EG$  is equal to  $AC$ , the side  $FG$  to  $BC$ , and the angle  $FGE$  to  $BCA$ ; which was to be proved.

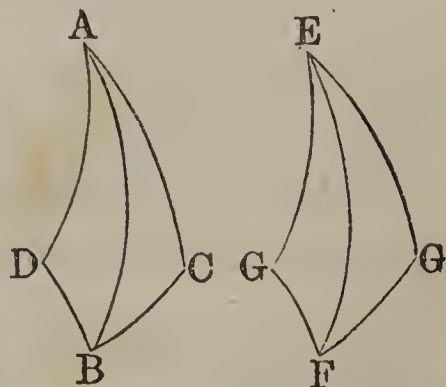


PROPOSITION X. THEOREM.

*If two spherical triangles on the same, or on equal spheres, have their sides equal, each to each, their angles will be equal, each to each, the equal angles lying opposite the equal sides.*

Let the spherical triangles  $EFG$  and  $ABC$  have the side  $EF$  equal to  $AB$ , the side  $EG$  equal to  $AC$ , and the side  $FG$  equal to  $BC$ : then will the angle  $FEG$  be equal to  $BAC$ , the angle  $EFG$  to  $ABC$ , and the angle  $EGF$  to  $ACB$ , and the equal angles will lie opposite the equal sides.

For, it may be shown by the same course of reasoning as that employed in B. I., P. X., that the triangle  $EFG$  is equal in all respects, either to the triangle  $ABC$ , or to its symmetrical triangle  $ADB$ : hence, the angle  $FEG$ , opposite to the side  $FG$ , is equal to the angle  $BAC$ ,





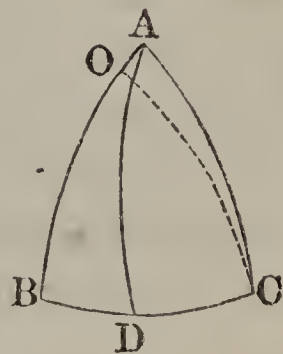
opposite to  $BC$ ; the angle  $EFG$ , opposite to  $EG$ , is equal to the angle  $ABC$ , opposite to  $AC$ ; and the angle  $EGF$ , opposite to  $EF$ , is equal to the angle  $ACB$ , opposite to  $AB$ ; which was to be proved.

PROPOSITION XI. THEOREM.

*In any isosceles spherical triangle, the angles opposite the equal sides are equal; and conversely, if two angles of a spherical triangle are equal, the triangle is isosceles.*

1°. Let  $ABC$  be a spherical triangle, having the side  $AB$  equal to  $AC$ : then will the angle  $C$  be equal to the angle  $B$ .

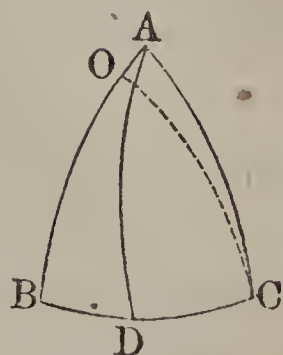
For; draw the arc of a great circle from the vertex  $A$ , to the middle point  $D$ , of the base  $BC$ : then in the two triangles  $ADB$  and  $ADC$ , we shall have the side  $AB$  equal to  $AC$ , by hypothesis, the side  $BD$  equal to  $DC$ , by construction, and the side  $AD$  common; consequently, the triangles have their angles equal, each to each (P. X.): hence, the angle  $C$  is equal to the angle  $B$ ; which was to be proved.



2°. Let  $ABC$  be a spherical triangle having the angle  $C$  equal to the angle  $B$ : then will the side  $AB$  be equal to the side  $AC$ , and consequently the triangle will be isosceles.

For, suppose that  $AB$  and  $AC$  are not equal, but that one of them, as  $AB$ , is the greater. On  $AB$  lay off the arc  $BO$  equal to  $AC$ , and draw the arc of a great circle from  $O$  to  $C$ : then in the triangles  $ACB$  and  $OBC$ , we shall have the side  $AC$  equal to  $OB$ , by construction,

the side  $BC$  common, and the included angle  $ACB$  equal to the included angle  $OBC$ , by hypothesis: hence, the remaining parts of the triangles are equal, each to each, and consequently, the angle  $OCB$  is equal to the angle  $ABC$ . But, the angle  $ACB$  is equal to  $ABC$ , by hypothesis, and therefore, the angle  $OCB$  is equal to  $ACB$ , or a part is equal to the whole, which is impossible: hence, the supposition that  $AB$  and  $AC$  are unequal, is absurd; they are therefore equal, and consequently, the triangle  $ABC$  is isosceles; *which was to be proved.*



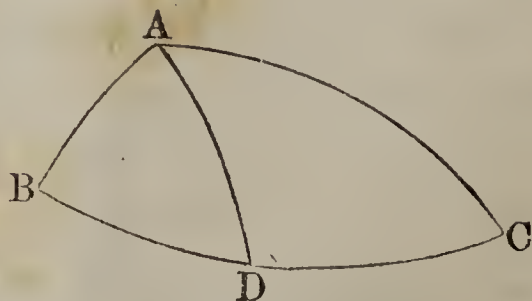
*Cor.* The triangles  $ADB$  and  $ADC$ , having all of their parts equal, each to each, the angle  $ADB$  is equal to  $ADC$ , and the angle  $DAB$  is equal to  $DAC$ ; that is, *if an arc of a great circle be drawn from the vertex of an isosceles spherical triangle to the middle of its base, it will be perpendicular to the base, and will bisect the vertical angle of the triangle.*

#### PROPOSITION XII. THEOREM.

*In any spherical triangle, the greater side is opposite the greater angle; and conversely, the greater angle is opposite the greater side.*

1°. Let  $ABC$  be a spherical triangle, in which the angle  $A$  is greater than the angle  $B$ : then will the side  $BC$  be greater than the side  $AC$ .

For, draw the arc  $AD$ , making the angle  $BAD$  equal to  $ABD$ : then will  $AD$  be equal to  $BD$  (P. XI.). But, the sum of  $AD$  and  $DC$  is



greater than  $AC$  (P. I.) ; or, putting for  $AD$  its equal  $BD$ , we have the sum of  $BD$  and  $DC$ , or  $BC$ , greater than  $AC$  ; *which was to be proved.*

2°. In the triangle  $ABC$ , let the side  $BC$  be greater than  $AC$  : then will the angle  $A$  be greater than the angle  $B$ .

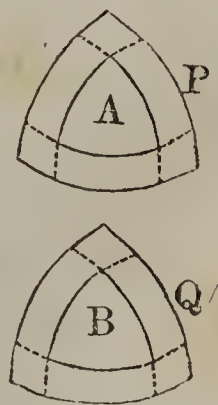
For, if the angles  $A$  and  $B$  were equal, the sides  $BC$  and  $AC$  would be equal ; or if the angle  $A$  was less than the angle  $B$ , the side  $BC$  would be less than  $AC$ , either of which conclusions is absurd : hence, the angle  $A$  is greater than the angle  $B$  ; *which was to be proved.*

#### PROPOSITION XIII. THEOREM.

*If two triangles on the same, or on equal spheres, are mutually equiangular, they are also mutually equilateral.*

Let the spherical triangles  $A$  and  $B$ , be mutually equiangular : then will they also be mutually equilateral.

For, let  $P$  be the polar triangle of  $A$ , and  $Q$  the polar triangle of  $B$  : then, because the triangles  $A$  and  $B$  are mutually equiangular, their polar triangles  $P$  and  $Q$ , must be mutually equilateral (P. VI.), and consequently mutually equiangular (P. X.). But, the triangles  $P$  and  $Q$  being mutually equiangular, their polar triangles  $A$  and  $B$ , are mutually equilateral (P. VI.) ; *which was to be proved.*



*Scholium.* This proposition does not hold good for plane triangles, for all similar plane triangles are mutually equiangular, but not necessarily mutually equilateral. Two spherical triangles on the same or on equal spheres, cannot be similar without being equal.

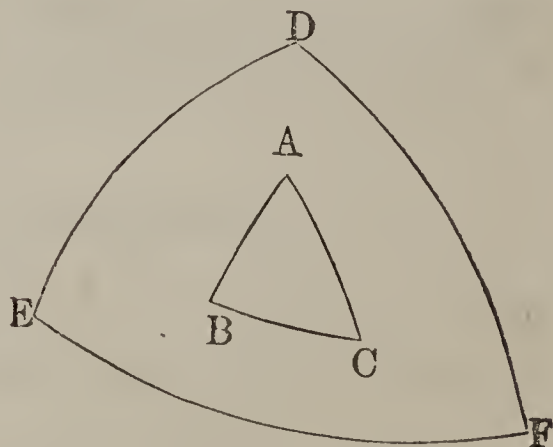


## PROPOSITION XIV. THEOREM.

*The sum of the angles of a spherical triangle is less than six right angles, and greater than two right angles.*

Let  $ABC$  be a spherical triangle, and  $DEF$  its polar triangle: then will the sum of the angles  $A$ ,  $B$ , and  $C$ , be less than six right angles and greater than two.

For, any angle, as  $A$ , being measured by a semi-circumference, minus the side  $EF$  (P. VI.), is less than two right angles: hence, the sum of the three angles is less than six right angles; and because the measure of each angle is equal to a semi-circumference, minus the side lying opposite

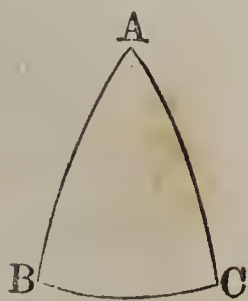


to it, in the polar triangle, the measure of the sum of the three angles is equal to three semi-circumferences, minus the sum of the sides of the polar triangle  $DEF$ . But the latter sum is less than a circumference; consequently, the measure of the sum of the angles  $A$ ,  $B$ , and  $C$ , is greater than a semi-circumference, and therefore the sum of the angles is greater than two right angles: hence, the sum of the angles  $A$ ,  $B$ , and  $C$ , is less than six right angles, and greater than two; *which was to be proved.*

*Cor. 1.* The sum of the three angles of a spherical triangle is not constant, like that of the angles of a rectilineal triangle, but varies between two right angles and six, without ever reaching either of these limits. Two angles, therefore, do not serve to determine the third.

*Cor. 2.* A spherical triangle may have two, or even three of its angles right angles; also two, or even three of its angles obtuse.

*Cor. 3.* If the triangle  $ABC$  is *bi-rectangular*, that is, has two right angles  $B$  and  $C$ , the opposite sides of the polar triangle will be quadrants, and their point of intersection will be the pole of the other side (P. III., C. 4). The angles opposite the equal sides are right angles (P. III., C. 3): hence, the sides  $AB$  and  $AC$  are quadrants.



If the angle  $A$  is also a right angle, the triangle  $ABC$  is *tri-rectangular*; each of its angles is a right angle, and its sides are quadrants. Four tri-rectangular triangles make up the surface of a hemisphere, and eight the entire surface of a sphere.

*Scholium.* The right angle is taken as the unit of measure of spherical angles, and is denoted by 1.

The excess of the sum of the angles of a spherical triangle over two right angles, is called the *spherical excess*. If we denote the spherical excess by  $E$ , and the three angles expressed in terms of the right angle, as a unit, by  $A$ ,  $B$ , and  $C$ , we shall have,

$$E = A + B + C - 2.$$

The *spherical excess* of any spherical polygon is equal to the excess of the sum of its angles over two right angles taken as many times as the polygon has sides, less two. If we denote the spherical excess by  $E$ , the sum of the angles by  $S$ , and the number of sides by  $n$ , we shall have,

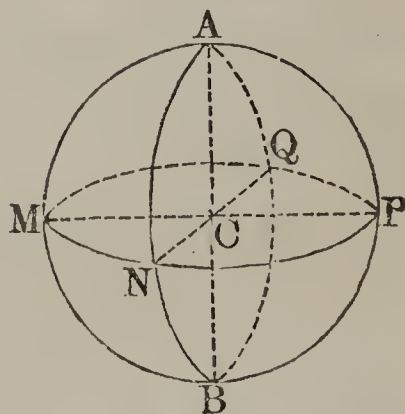
$$E = S - 2(n - 2) = S - 2n + 4.$$

## PROPOSITION XV. THEOREM.

*Any lune, is to the surface of the sphere, as the angle of the lune is to four right angles, or as the arc which measures that angle is to the circumference of a great circle.*

Let  $AMBN$  be a lune, and  $MCN$  the angle of the lune then will the area of the lune be to the surface of the sphere, as the arc  $MN$  is to the circumference of a great circle  $MNPQ$ ; or, which is the same thing, as the angle  $MCN$  is to four right angles.

In the first place, suppose the arc  $MN$  and the circumference  $MNPQ$  to be commensurable. For example, let them be to each other as 5 is to 48. Divide the circumference  $MNPQ$  into 48 equal parts, beginning at  $M$ ;  $MN$  will contain five of these parts. Join each point of division with the points  $A$  and  $B$ , by a quadrant: there will be formed 96 equal isosceles spherical triangles (P. VII., S. 2) on the surface of the sphere, of which the lune will contain 10: hence, in this case, the area of the lune is to the surface of the sphere, as 10 is to 96, or as 5 is to 48; that is, as the arc  $MN$  is to the circumference  $MNPQ$ , or as the angle of the lune is to four right angles.



In like manner, the same relation may be shown to exist when the arc  $MN$ , and the circumference  $MNPQ$ , are to each other as any other whole numbers.

If the arc  $MN$ , and the circumference  $MNPQ$ , are not commensurable, the same relation may be shown to exist by



a course of reasoning entirely analogous to that employed in Book IV., Proposition III. Hence, in all cases, the area of a lune is to the surface of the sphere, as the angle of the lune is to four right angles, or as the arc which measures that angle is to the circumference of a great circle; *which was to be proved.*

*Cor. 1.* Lunes, on the same or on equal spheres, are to each other as their angles.

*Cor. 2.* If we denote the area of a tri-rectangular triangle by  $T$ , the area of a lune by  $L$ , and the angle of the lune by  $A$ , the right angle being denoted by 1, we shall have,

$$L : 8T :: A : 4;$$

whence,

$$L = T \times 2A;$$

hence, the area of a lune is equal to the area of a tri-rectangular triangle multiplied by twice the angle of the lune.

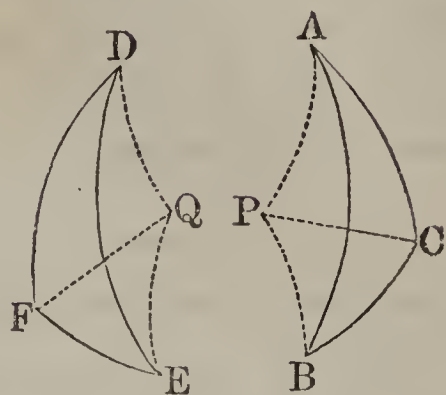
*Scholium.* The spherical wedge, whose angle is  $MCN$ , is to the entire sphere, as the angle of the wedge is to four right angles, as may be shown by a course of reasoning entirely analogous to that just employed: hence, we infer that the volume of a spherical wedge is equal to the lune which forms its base, multiplied by one-third of the radius.

#### PROPOSITION XVI. THEOREM.

*Symmetrical triangles are equal in area.*

Let  $ABC$  and  $DEF$  be symmetrical triangles, the side  $DE$  being equal to  $AB$ , the side  $DF$  to  $AC$ , and the side  $EF$  to  $BC$ : then will the triangles be equal in area.

For, conceive a small circle to be drawn through  $A$ ,  $B$ , and  $C$ , and let  $P$  be its pole; draw arcs of great circles from  $P$  to  $A$ ,  $B$ , and  $C$ : these arcs will be equal (D. 7). Draw the arc of a great circle  $FQ$ , making the angle  $DFQ$  equal to  $ACP$ , and lay off on it,  $FQ$  equal to  $CP$ ; draw arcs of great circles  $QD$  and  $QE$ .



In the triangles  $PAC$  and  $FDQ$ , we have the side  $FD$  equal to  $AC$ , by hypothesis; the side  $FQ$  equal to  $PC$ , by construction, and the angle  $DFQ$  equal to  $ACP$ , by construction: hence (P. VIII.), the side  $DQ$  is equal to  $AP$ , the angle  $FDQ$  to  $PAC$ , and the angle  $FQD$  to  $APC$ . Now, because the triangles  $QFD$  and  $PAC$  are isosceles and equal in all their parts, they may be placed so as to coincide throughout, the side  $DF$  falling on  $AC$ , and the side  $QD$  on  $PA$ : hence, they are equal in area.

If we take, from the angle  $DFE$  the angle  $DFQ$ , and from the angle  $ACB$  the angle  $ACP$ , the remaining angles  $QFE$  and  $PCB$ , will be equal. In the triangles  $FQE$  and  $PCB$ , we have the side  $QF$  equal to  $PC$ , by construction, the side  $FE$  equal to  $BC$ , by hypothesis, and the angle  $QFE$  equal to  $PCB$ , from what has just been shown; hence, the triangles are equal in all their parts, and being isosceles, they may be placed so as to coincide throughout, the side  $QE$  falling on  $PB$ , and the side  $QF$  on  $PC$ ; these triangles are, therefore, equal in area.

In the triangles  $QDE$  and  $PAB$ , we have the sides  $QD$ ,  $QE$ ,  $PA$ , and  $PB$ , all equal, and the angle  $DQE$  equal to  $APB$ , because they are the sums of equal angles: hence, the triangles are equal in all their parts, and

because they are isosceles, they may be so placed as to coincide throughout, the side  $QD$  falling on  $PB$ , and the side  $QE$  on  $PA$ ; these triangles are, therefore, equal in area.

Hence, the sum of the triangles  $QFD$  and  $QFE$ , is equal to the sum of the triangles  $PAC$  and  $PBC$ . If from the former sum we take away the triangle  $QDE$ , there will remain the triangle  $DFE$ ; and if from the latter sum we take away the triangle  $PAB$ , there will remain the triangle  $ABC$ : hence, the triangles  $ABC$  and  $DEF$  are equal in area; *which was to be proved.*

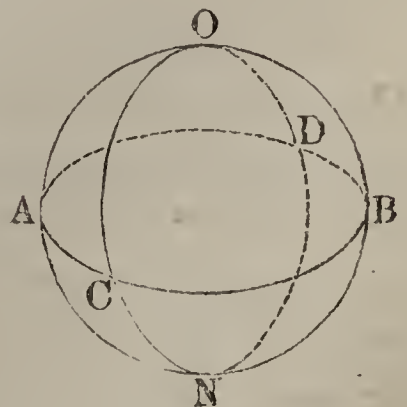
*Scholium.* If the point  $P$  falls within the triangle  $ABC$ , the point  $Q$  will fall within the triangle  $DEF$ . In this case, the triangle  $DEF$  is equal to the sum of the triangles  $QFD$ ,  $QFE$ , and  $QDE$ , and the triangle  $ABC$  is equal to the sum of the equal triangles  $PAC$ ,  $PBC$ , and  $PAB$ ; the proposition, therefore, still holds good.

PROPOSITION XVII. THEOREM.

*If the circumferences of two great circles intersect on the surface of a hemisphere, the sum of the opposite triangles thus formed, is equal to a lune whose angle is equal to that formed by the circles.*

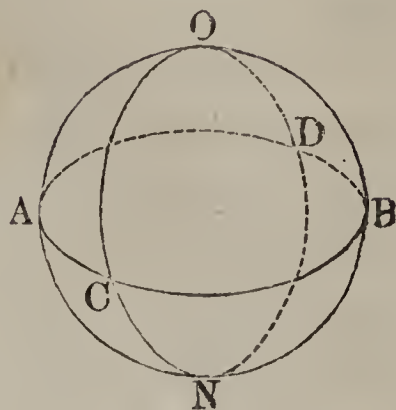
Let the circumferences  $AOB$ ,  $COD$ , intersect on the surface of a hemisphere: then will the sum of the opposite triangles  $AOC$ ,  $BOD$ , be equal to the lune whose angle is  $BOD$ .

For, produce the arcs  $OB$ ,  $OD$ , on the other hemisphere, till they meet at  $N$ . Now, since  $AOB$  and  $ONB$  are semi-circumferences, if we take away the common part





$OB$ , we shall have  $BN$  equal to  $AO$ . For a like reason, we have  $DN$  equal to  $CO$ , and  $BD$  equal to  $AC$ : hence, the two triangles  $AOC$ ,  $BDN$ , have their sides respectively equal: they are therefore symmetrical; consequently, they are equal in area (P. XVI.). But the sum of the triangles  $BDN$ ,  $BOD$ , is equal to the lune  $OBND$ , whose angle is  $BOD$ : hence, the sum of  $AOC$  and  $BOD$  is equal to the lune whose angle is  $BOD$ ; which was to be proved.



*Scholium.* It is evident that the two spherical pyramids, which have the triangles  $AOC$ ,  $BOD$ , for bases, are together equal to the spherical wedge whose angle is  $BOD$ .

#### PROPOSITION XVIII. THEOREM.

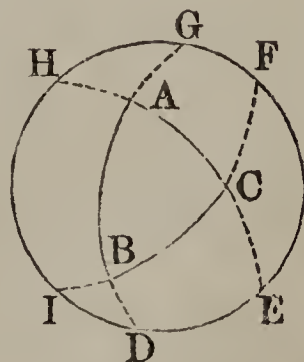
*The area of a spherical triangle is equal to its spherical excess multiplied by a tri-rectangular triangle.*

Let  $ABC$  be a spherical triangle: then will its surface be equal to

$$(A + B + C - 2) \times T.$$

For, produce its sides till they meet the great circle  $DEFG$ , drawn at pleasure, without the triangle. By the last theorem, the two triangles  $ADE$ ,  $AGH$ , are together equal to the lune whose angle is  $A$ ; but the area of this lune is equal to  $2A \times T$  (P. XV., C. 2):

hence, the sum of the triangles  $ADE$  and  $AGH$ , is equal to  $2A \times T$ . In like manner, it may be shown that the



sum of the triangles  $BFG$  and  $BID$ , is equal to  $2B \times T$ , and that the sum of the triangles  $CIH$  and  $CFE$ , is equal to  $2C \times T$ .

But the sum of these six triangles exceeds the hemisphere, or four times  $T$ , by twice the triangle  $ABC$ . We shall therefore have,

$$2 \times \text{area } ABC = 2A \times T + 2B \times T + 2C \times T - 4T;$$

or, by reducing and factoring,

$$\text{area } ABC = (A + B + C - 2) \times T;$$

which was to be proved.

*Scholium 1.* The same relation which exists between the spherical triangle  $ABC$ , and the tri-rectangular triangle, exists also between the spherical pyramid which has  $ABC$  for its base, and the tri-rectangular pyramid. The triedral angle of the pyramid is to the triedral angle of the tri-rectangular pyramid, as the triangle  $ABC$  to the tri-rectangular triangle. From these relations, the following consequences are deduced:

1°. Triangular spherical pyramids are to each other as their bases; and since a polygonal pyramid may always be divided into triangular pyramids, it follows that any two spherical pyramids are to each other as their bases.

2°. Polyedral angles at the centre of the same, or of equal spheres, are to each other as the spherical polygons intercepted by their faces.

*Scholium 2.* A polyedral angle whose faces are perpendicular to each other, is called a *right polyedral angle*; and if placed at the centre of a sphere, its faces will intercept a tri-rectangular triangle. The right polyedral angle is

taken as the unit of polyedral angles, and the tri-rectangular spherical triangle is taken as its measure. If the vertex of a polyedral angle be taken as the centre of a sphere, the portion of the surface intercepted by its faces will be the measure of the polyedral angle, a tri-rectangular triangle of the same sphere, being the unit.

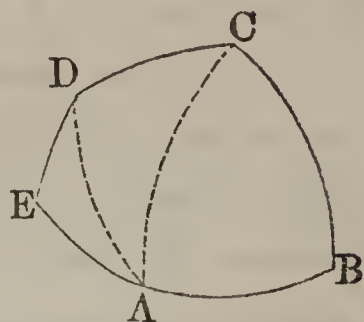
PROPOSITION XIX. THEOREM.

*The area of a spherical polygon is equal to its spherical excess multiplied by the tri-rectangular triangle.*

Let  $ABCDE$  be a spherical polygon, the sum of whose angles is  $S$ , and the number of whose sides is  $n$  : then will its area be equal to

$$(S - 2n + 4) \times T.$$

For, draw the diagonals  $AC$ ,  $AD$ , dividing the polygon into spherical triangles : there will be  $n - 2$  such triangles. Now, the area of each triangle is equal to its spherical excess into the tri-rectangular triangle : hence, the sum of the areas of all the triangles, or the area of the polygon, is equal to the sum of all the angles of the triangles, or the sum of the angles of the polygon diminished by  $2(n - 2)$  into the tri-rectangular triangle ; or,



$$\text{area } ABCDE = [S - 2(n - 2)] \times T ;$$

whence, by reduction,

$$\text{area } ABCDE = (S - 2n + 4) \times T ;$$

*which was to be proved.*



## GENERAL SCHOLIUM.

From any point on a hemisphere, two arcs of great circles can always be drawn which shall be perpendicular to the circumference of the base of the hemisphere, and they will in general be unequal. Now, it may be proved, by a course of reasoning analogous to that employed in Book I., Proposition XV.:

1°. That the shorter of the two arcs is the shortest arc that can be drawn from the given point to the circumference.

2°. That two oblique arcs drawn from the same point, to points of the circumference at equal distances from the foot of the perpendicular, are equal:

3°. That of two oblique arcs, that is the longer which meets the circumference at the greater distance from the foot of the perpendicular.

This property of the sphere is used in the discussion of triangles in spherical trigonometry.

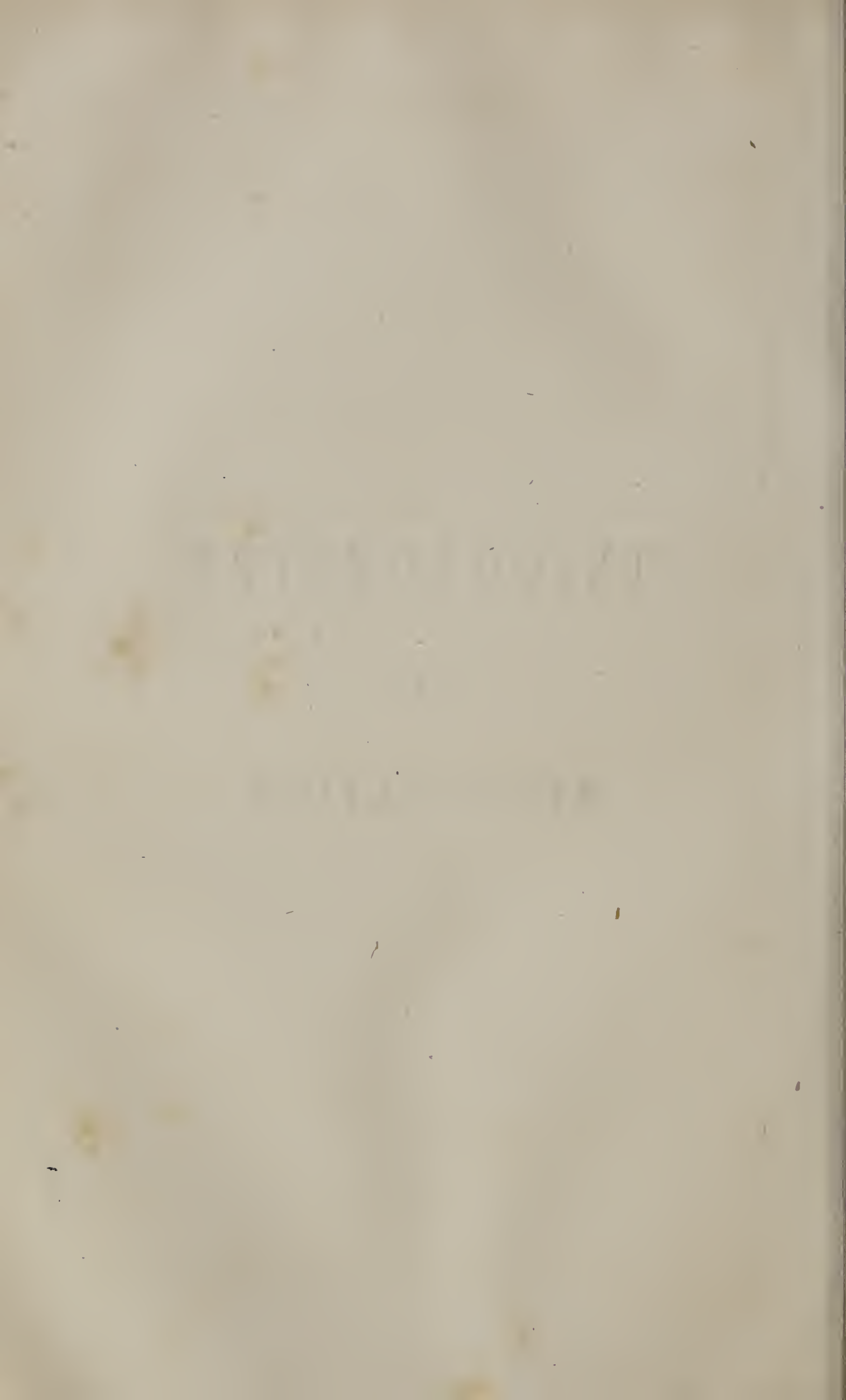


TRIGONOMETRY

AND

MENSURATION.





# INTRODUCTION TO TRIGONOMETRY.

---

## LOGARITHMS.

1. THE LOGARITHM of a number is the exponent of the power to which it is necessary to raise a fixed number, to produce the given number.

The fixed number is called the *base of the system*. Any positive number, except 1, may be taken as the base of a system. In the common system, the base is 10.

2. If we denote any positive number by  $n$ , and the corresponding exponent of 10, by  $x$ , we shall have the exponential equation,

$$10^x = n. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.)$$

In this equation,  $x$  is, by definition, the logarithm of  $n$ , which may be expressed thus,

$$x = \log n. \quad . \quad . \quad . \quad . \quad . \quad (2.)$$

3. From the definition of a logarithm, it follows that, *the logarithm of any power of 10 is equal to the exponent of that power*: hence the formula,

$$\log (10)^p = p. \quad . \quad . \quad . \quad . \quad . \quad (3.)$$

If a number is an exact power of 10, its logarithm is a whole number.

If a number is not an exact power of 10, its logarithm will not be a whole number, but will be made up of *an entire part* plus a *fractional part*, which is generally expressed decimally. The entire part of a logarithm is called *the characteristic*, the decimal part, is called the *mantissa*.

4. If, in Equation (3), we make  $p$  successively equal to 0, 1, 2, 3, &c., and also equal to  $-0$ ,  $-1$ ,  $-2$ ,  $-3$ , &c., we may form the following

TABLE.

log 1	=	0		
log 10	=	1	log .1	= - 1
log 100	=	2	log .01	= - 2
log 1000	=	3	log .001	= - 3
&c., &c.			&c., &c.	

If a number lies between 1 and 10, its logarithm lies between 0 and 1, that is, it is equal to 0 *plus* a decimal; if a number lies between 10 and 100, its logarithm is equal to 1 *plus* a decimal; if between 100 and 1000, its logarithm is equal to 2 *plus* a decimal; and so on: hence, we have the following

## RULE.

*The characteristic of the logarithm of an entire number is positive, and numerically 1 less than the number of places of figures in the given number.*

If a decimal fraction lies between .1 and 1, its logarithm lies between  $-1$  and 0, that is, it is equal to  $-1$  *plus* a decimal; if a number lies between .01 and .1, its logarithm is equal to  $-2$ , *plus* a decimal; if between .001 and .01, its logarithm is equal to  $-3$ , *plus* a decimal; and so on: hence, the following



RULE.

*The characteristic of the logarithm of a decimal fraction is negative, and numerically 1 greater than the number of 0's that immediately follow the decimal point.*

The characteristic alone is negative, *the mantissa being always positive.* This fact is indicated by writing the negative sign over the characteristic: thus,  $\overline{2}.371465$ , is equivalent to  $-2 + .371465$ .

It is to be observed, that the characteristic of the logarithm of a mixed number is the same as that of its entire part. Thus, the mixed number 74.103, lies between 10 and 100; hence, its logarithm lies between 1 and 2, as does the logarithm of 74.

GENERAL PRINCIPLES.

5. Let  $m$  and  $n$  denote any two numbers, and  $x$  and  $y$  their logarithms. We shall have, from the definition of a logarithm, the following equations,

$$10^x = m. \quad . \quad . \quad . \quad . \quad . \quad (4.)$$

$$10^y = n. \quad . \quad . \quad . \quad . \quad . \quad (5.)$$

Multiplying (4) and (5), member by member, we have,

$$10^{x+y} = mn;$$

whence, by the definition,

$$x + y = \log(mn). \quad . \quad . \quad . \quad . \quad (6.)$$

That is, *the logarithm of the product of two numbers is equal to the sum of the logarithms of the numbers.*

6. Dividing (4) by (5), member by member, we have,

$$10^{x-y} = \frac{m}{n};$$

whence, by the definition,

$$x - y = \log \left( \frac{m}{n} \right). \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (7.)$$

That is, *the logarithm of a quotient is equal to the logarithm of the dividend diminished by that of the divisor.*

7. Raising both members of (4) to the power denoted by  $p$ , we have,

$$10^{xp} = m^p;$$

whence, by the definition,

$$xp = \log m^p. \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (8.)$$

That is, *the logarithm of any power of a number is equal to the logarithm of the number multiplied by the exponent of the power.*

8. Extracting the root, indicated by  $r$ , of both members of (4), we have,

$$10^{\frac{x}{r}} = \sqrt[r]{m};$$

whence, by the definition,

$$\frac{x}{r} = \log \sqrt[r]{m}. \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (9.)$$

That is, *the logarithm of any root of a number is equal to the logarithm of the number divided by the index of the root.*

The preceding principles enable us to abbreviate the operations of multiplication and division, by converting them into the simpler ones of addition and subtraction.

## TABLE OF LOGARITHMS.

9. A TABLE OF LOGARITHMS, is a table by means of which we can find the logarithm corresponding to any number, or the number corresponding to any logarithm.

In the table appended, the complete logarithm is given for all numbers from 1 up to 100. For other numbers, the mantissas alone are given; the characteristic may be found by one of the rules of Art. 4.

Before explaining the use of the table, it is to be shown that the mantissa of the logarithm of any number is not changed by multiplying or dividing the number by any exact power of 10.

Let  $n$  represent any number whatever, and  $10^p$  any power of 10,  $p$  being any whole number, either positive or negative. Then, in accordance with the principles of Arts. 5 and 3, we shall have,

$$\log (n \times 10^p) = \log n + \log 10^p = p + \log n ;$$

but  $p$  is, by hypothesis, a whole number: hence, the decimal part of the  $\log (n \times 10^p)$  is the same as that of  $\log n$ ; *which was to be proved.*

Hence, in finding the mantissa of the logarithm of a number, we may regard the number as a decimal, and move the decimal point to the right or left, at pleasure. Thus, the mantissa of the logarithm of 456357, is the same as that of the number 4563.57; and the mantissa of the logarithm of 2.00357, is the same as that of 2003.57.



## MANNER OF USING THE TABLE.

1°. *To find the logarithm of a number less than 100.*

10. Look on the first page, in the column headed "N," for the given number; the number opposite is the logarithm required. Thus,

$$\log 67 = 1.826075.$$

2°. *To find the logarithm of a number between 100 and 10,000.*

11. Find the characteristic by the first rule of Art. 4.

To find the mantissa, look in the column headed "N," for the first three figures of the number; then pass along a horizontal line until you come to the column headed with the fourth figure of the number; at this place will be found four figures of the mantissa, to which, two other figures, taken from the column headed "0," are to be prefixed. If the figures found stand opposite a row of six figures, in the column headed "0," the first two of this row are the ones to be prefixed; if not, ascend the column till a row of six figures is found; the first two, of this row, are the ones to be prefixed.

If, however, in passing back from the four figures, first found, any *dots* are passed, the two figures to be prefixed must be taken from the line immediately below. If the figures first found fall at a place where dots occur, the dots must be replaced by 0's, and the figures to be prefixed must be taken from the *line below*. Thus,

$$\text{Log } 8979 = 3.953228$$

$$\text{Log } 3098 = 3.491081$$

$$\text{Log } 2188 = 3.340047$$

3°. To find the logarithm of a number greater than 10,000.

12. Find the characteristic by the first rule of Art. 4.

To find the mantissa, place a decimal point after the fourth figure (Art. 9), thus converting the number into a mixed number. Find the mantissa of the entire part, by the method last given. Then take from the column headed "D," the corresponding *tabular difference*, and multiply this by the decimal part and add the product to the mantissa just found. The result will be the required mantissa.

It is to be observed that when the decimal part of the product just spoken of is equal to or exceeds .5, we add 1 to the entire part, otherwise the decimal part is rejected.

#### EXAMPLE.

1. To find the logarithm of 672887.

The characteristic is 5. Placing a decimal point after the fourth figure, the number becomes 6728.87. The mantissa of the logarithm of 6728 is 827886, and the corresponding number in the column "D" is 65. Multiplying 65 by .87, we have 56.55; or, since the decimal part exceeds .5, 57. We add 57 to the mantissa already found, giving 827943, and we finally have,

$$\log 672887 = 5.827943.$$

The numbers in the column "D" are the differences between the logarithms of two consecutive whole numbers, and are found by subtracting the number under the heading "4" from that under the heading "5."

In the example last given, the mantissa of the logarithm of 6728 is 827886, and that of 6729 is 827951, and their difference is 65; 87 hundredths of this difference is

57 : hence, the mantissa of the logarithm of 6728.87 is found by adding 57 to 827886. The principle employed is, that the differences of numbers are proportional to the differences of their logarithms, when these differences are small.

4°. *To find the logarithm of a decimal.*

13. Find the characteristic by the second rule of Art. 4.

To find the mantissa, drop the decimal point, thus reducing the decimal to a whole number. Find the mantissa of the logarithm of this number, and it will be the mantissa required. Thus,

$$\log .0327 = \bar{2}.514548$$

$$\log 378.024 = 2.577520$$

5°. *To find the number corresponding to a given logarithm.*

14. The rule is the reverse of those just given. Look in the table for the mantissa of the given logarithm. If it cannot be found, take out the next less mantissa, and also the corresponding number, which set aside. Find the difference between the mantissa taken out and that of the given logarithm ; annex as many 0's as may be necessary, and divide this result by the corresponding number in the column "D." Annex the quotient to the number set aside, and then point off, from the left hand, a number of places of figures equal to the characteristic plus 1 : the result will be the number required. If the characteristic is negative, the result will be a pure decimal, and the number of 0's which immediately follow the decimal point will be one less than the number of units in the characteristic.



EXAMPLES.

1. Let it be required to find the number corresponding to the logarithm 5.233568.

The next less mantissa in the table is 233504; the corresponding number is 1712, and the tabular difference is 253.

OPERATION.

Given mantissa,	. . . . .	233568	
Next less mantissa,	. . . . .	233504	. . . 1712
		253	) 6400000 ( 25296

∴ The required number is 171225.296.

The number corresponding to the logarithm  $\overline{2}.233568$  is .0171225.

2. What is the number corresponding to the logarithm  $\overline{2}.785407$  ? Ans. .06101084.

3. What is the number corresponding to the logarithm  $\overline{1}.846741$  ? Ans. .702653.

MULTIPLICATION BY MEANS OF LOGARITHMS.

15. From the principle proved in Art. 5, we deduce the following

RULE.

*Find the logarithms of the factors, and take their sum; then find the number corresponding to the resulting logarithm, and it will be the product required.*

## INTRODUCTION.

## EXAMPLES.

1. Multiply 23.14 by 5.062.

## OPERATION.

$$\begin{array}{rcl}
 \log 23.14 & . & . & . & 1.364363 \\
 \log 5.062 & . & . & . & 0.704322 \\
 \hline
 & & & & 2.068685 \quad \therefore 117.1347, \text{ product.} \\
 \hline
 \end{array}$$

2. Find the continued product of 3.902, 597.16, and 0.0314728.

## OPERATION.

$$\begin{array}{rcl}
 \log 3.902 & . & . & . & 0.591287 \\
 \log 597.16 & . & . & . & 2.776091 \\
 \log 0.0314728 & . & . & . & \overline{2.497936} \\
 \hline
 & & & & 1.865314 \quad \therefore 73.3354, \text{ product.} \\
 \hline
 \end{array}$$

Here, the  $\overline{2}$  cancels the  $+2$ , and the 1 carried from the decimal part is set down.

3. Find the continued product of 3.586, 2.1046, 0.8372, and 0.0294. *Ans.* 0.1857615.

## DIVISION BY MEANS OF LOGARITHMS.

16. From the principle proved in Art. 6, we have the following

## RULE.

*Find the logarithms of the dividend and divisor, and subtract the latter from the former; then find the number corresponding to the resulting logarithm, and it will be the quotient required.*

EXAMPLES.

1. Divide 24163 by 4567.

OPERATION.

$$\begin{array}{rcl} \log 24163 & . & . & . & 4.383151 \\ \log 4567 & . & . & . & 3.659631 \\ \hline & & & & 0.723520 & \therefore 5.29078, \text{ quotient.} \end{array}$$

2. Divide 0.7438 by 12.9476.

OPERATION.

$$\begin{array}{rcl} \log 0.7438 & . & . & . & \bar{1}.871456 \\ \log 12.9476 & . & . & . & 1.112189 \\ \hline & & & & \bar{2}.759267 & \therefore 0.057447, \text{ quotient.} \end{array}$$

Here, 1 taken from  $\bar{1}$ , gives  $\bar{2}$  for a result. The subtraction, as in this case, is always to be performed in the algebraic sense.

3. Divide 37.149 by 523.76.

*Ans.* 0.0709274.

The operation of division, particularly when combined with that of multiplication, can often be simplified by using the principle of

THE ARITHMETICAL COMPLEMENT.

17. The ARITHMETICAL COMPLEMENT of a logarithm is the result obtained by subtracting it from 10. Thus, 8.130456 is the arithmetical complement of 1.869544. The arithmetical complement of a logarithm may be written out *by commencing at the left hand and subtracting each figure from 9,*



until the last significant figure is reached, which must be taken from 10. The arithmetical complement is denoted by the symbol (a. c.).

Let  $a$  and  $b$  represent any two logarithms whatever, and  $a - b$  their difference. Since we may add 10 to, and subtract it from,  $a - b$ , without altering its value, we have,

$$a - b = a + (10 - b) - 10. \quad . \quad . \quad . \quad (10.)$$

But,  $10 - b$  is, by definition, the arithmetical complement of  $b$ : hence, Equation (10) shows that the difference between two logarithms is equal to *the first, plus the arithmetical complement of the second, minus 10*.

Hence, to divide one number by another by means of the arithmetical complement, we have the following

#### RULE.

*Find the logarithm of the dividend, and the arithmetical complement of the logarithm of the divisor, add them together, and diminish the sum by 10; the number corresponding to the resulting logarithm will be the quotient required.*

#### EXAMPLES.

1. Divide 327.5 by 22.07.

#### OPERATION.

log 327.5	. . .	2.515211	
(a. c.) log 22.07	. . .	8.656198	
		1.171409	∴ 14.839, quotient.

2. Divide 37.149 by 523.76.

*Ans.* 0.0709273.

3. Multiply 358884 by 5672, and divide the product by 89721.

OPERATION.

log 358884	. . .	5.554954	
log 5672	. . .	3.753736	
(a. c.) log 89721	. . .	5.047106	
		<hr/>	
		4.355796	∴ 22688, result.
		<hr/>	

4. Solve the proportion,

$$3976 : 7952 :: 5903 : x.$$

OPERATION.

log 7952	. . .	3.900476	
log 5903	. . .	3.771073	
(a. c.) log 3976	. . .	6.400554	
		<hr/>	
		4.072103	∴ $x = 11806$
		<hr/>	

The operation of subtracting 10 is always performed mentally.

#### RAISING TO POWERS BY MEANS OF LOGARITHMS.

18. From the principle proved in Art. 7, we have the following

RULE.

*Find the logarithm of the number, and multiply it by the exponent of the power; then find the number corresponding to the resulting logarithm, and it will be the power required.*

## EXAMPLES.

1. Find the 5th power of 9.

## OPERATION.

$$\begin{array}{r} \log 9 \quad . \quad . \quad . \quad 0.954243 \\ \hline \phantom{\log 9 \quad . \quad . \quad . \quad } 5 \\ \hline 4.771215 \end{array} \quad \therefore \quad 59049, \text{ power.}$$

2. Find the 7th power of 8. *Ans.* 2097152.

## EXTRACTING ROOTS BY MEANS OF LOGARITHMS.

19. From the principle proved in Art. 8, we have the following

## RULE.

*Find the logarithm of the number, and divide it by the index of the root ; then find the number corresponding to the resulting logarithm, and it will be the root required.*

## EXAMPLES.

1. Find the cube root of 4096.

The logarithm of 4096 is 3.612360, and one-third of this is 1.204120. The corresponding number is 16, which is the root sought.

*When the characteristic is negative and not divisible by the index, add to it the smallest negative number that will make it divisible, and then prefix the same number, with a plus sign, to the mantissa.*

2. Find the 4th root of .00000081.

The logarithm of .00000081 is  $\overline{7}.908485$ , which is equal to  $\overline{8} + 1.908485$ , and one-fourth of this is  $\overline{2}.477121$ .

The number corresponding to this logarithm is .03 : hence, .03 is the root required.



# PLANE TRIGONOMETRY.

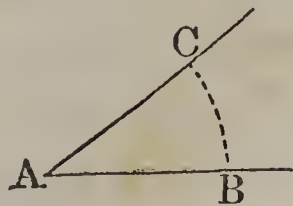
---

20 PLANE TRIGONOMETRY is that branch of Mathematics which treats of the *solution* of plane triangles.

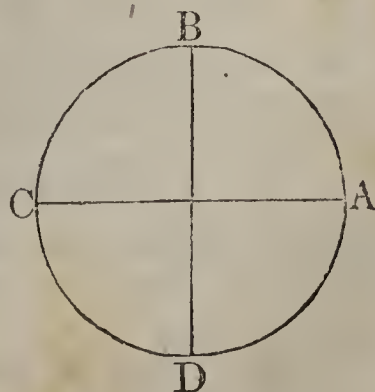
In every plane triangle there are six parts : *three sides* and *three angles*. When three of these parts are given, one being a side, the remaining parts may be found by computation. The operation of finding the unknown parts, is called the *solution* of the triangle.

21. A plane angle is measured by the arc of a circle included between its sides, the centre of the circle being at the vertex, and its radius being equal to 1.

Thus, if the vertex  $A$  be taken as a centre, and the radius  $AB$  be equal to 1, the intercepted arc  $BC$  will measure the angle  $A$  (B. III., P. XVII., S.).



Let  $ABCD$  represent a circle whose radius is equal to 1, and  $AC$ ,  $BD$ , two diameters perpendicular to each other. These diameters divide the circumference into four equal parts, called *quadrants*; and because each of the angles at the centre is a right angle, it follows that a *right angle* is measured by a *quad-*



*rant.* An *acute angle* is measured by an arc less than a quadrant, and an *obtuse angle*, by an arc greater than a quadrant.

22. In Geometry, the unit of angular measure is a *right angle*; so in Trigonometry, the *primary unit* is a quadrant, which is the measure of a right angle.

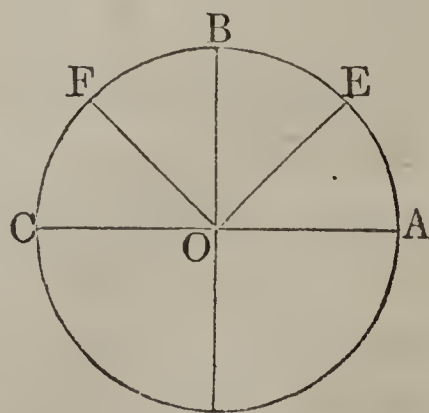
For convenience, the quadrant is divided into 90 equal parts, each of which is called a *degree*; each degree into 60 equal parts, called *minutes*; and each minute into 60 equal parts, called *seconds*. Degrees, minutes, and seconds, are denoted by the symbols  $^{\circ}$ ,  $'$ ,  $''$ . Thus, the expression  $7^{\circ} 22' 33''$ , is read, 7 *degrees*, 22 *minutes*, and 33 *seconds*. Fractional parts of a second are expressed decimally.

A quadrant contains 324,000 seconds, and an arc of  $7^{\circ} 22' 33''$  contains 26553 seconds; hence, the angle measured by the latter arc, is the  $\frac{26553}{324000}$ th part of a right angle. In like manner, any angle may be expressed in terms of a right angle.

23. The *complement* of an arc is the difference between that arc and  $90^{\circ}$ . The *complement* of an *angle* is the difference between that angle and a right angle.

Thus,  $EB$  is the complement of  $AE$ , and  $FB$  is the complement of  $AF$ . In like manner,  $EOB$  is the complement of  $AOE$ , and  $FOB$  is the complement of  $AOF$ .

In a right-angled triangle, the acute angles are complements of each other.



24. The *supplement* of an arc is the difference between

that arc and  $180^\circ$ . The *supplement of an angle* is the difference between that angle and two right angles.

Thus,  $EC$  is the supplement of  $AE$ , and  $FC$  the supplement of  $AF$ . In like manner,  $EOC$  is the supplement of  $AOE$ , and  $FOC$  the supplement of  $AOF$ .

In any plane triangle, either angle is the supplement of the sum of the other two.

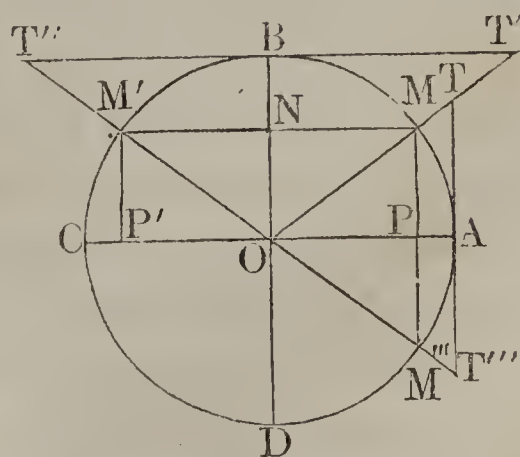
25. Instead of employing the arcs themselves, we usually employ certain *functions* of the arcs, as explained below. A *function* of a quantity is something which depends upon that quantity for its value.

The following functions are the only ones needed for solving triangles :

26. The *sine* of an arc is the distance of one extremity of the arc from the diameter, through the other extremity.

Thus,  $PM$  is the sine of  $AM$ , and  $P'M'$  is the sine of  $AM'$ .

If  $AM$  is equal to  $M'C$ ,  $AM$  and  $AM'$  will be supplements of each other ; and because  $MM'$  is parallel to  $AC$ ,  $PM$  will be equal to  $P'M'$  (B. I., P. XXIII.) : hence, *the sine of an arc is equal to the sine of its supplement*.



27. The *cosine* of an arc is the sine of the complement of the arc.

Thus,  $NM$  is the cosine of  $AM$ , and  $NM'$  is the cosine of  $AM'$ . These lines are respectively equal to  $OP$  and  $OP'$ .

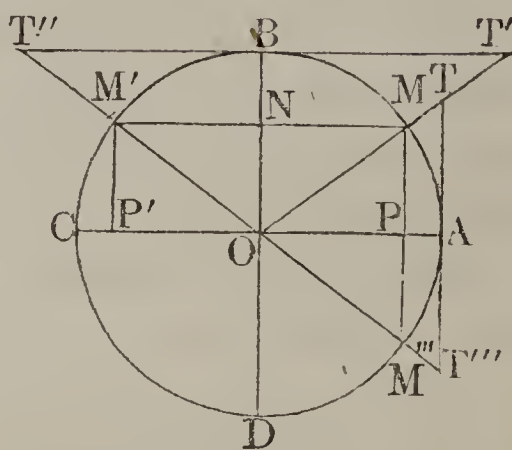


It is evident, from the equal triangles of the figure, that the cosine of an arc is equal to the cosine of its supplement.

28. The *tangent* of an arc is the perpendicular to the radius at one extremity of the arc, limited by the prolongation of the diameter through the other extremity

Thus,  $AT$  is the tangent of the arc  $AM$ , and  $AT'''$  is the tangent of the arc  $AM'$ .

If  $AM$  is equal to  $M'C$ ,  $AM$  and  $AM'$  will be supplements of each other. But  $AM'''$  and  $AM'$  are also supplements of each other: hence, the arc  $AM$  is equal to the arc  $AM'''$ , and the corresponding angles,  $AOM$  and  $AOM'''$ , are also equal. The right-angled triangles  $AOT$  and  $AOT'''$ , have a common base  $AO$ , and the angles at the base equal; consequently, the remaining parts are respectively equal: hence,  $AT$  is equal to  $AT'''$ . But  $AT$  is the tangent of  $AM$ , and  $AT'''$  is the tangent of  $AM'$ : hence, *the tangent of an arc is equal to the tangent of its supplement*.



It is to be observed that no account is taken of the algebraic signs of the cosines and tangents, the numerical values alone being referred to.

29. The *cotangent* of an arc is the tangent of its complement.

Thus,  $BT'$  is the cotangent of the arc  $AM$ , and  $BT''$  is the cotangent of the arc  $AM'$ .

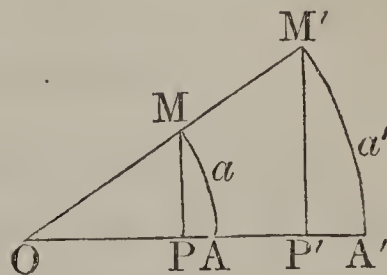
The sine, cosine, tangent, and cotangent of an arc,  $a$ , are, for convenience, written  $\sin a$ ,  $\cos a$ ,  $\tan a$ , and  $\cot a$ .

These functions of an arc have been defined on the supposition that the radius of the arc is equal to 1; in this case, they may also be considered as functions of the angle which the arc measures.

Thus,  $PM$ ,  $NM$ ,  $AT$ , and  $BT'$ , are respectively the sine, cosine, tangent, and cotangent of the angle  $AOM$ , as well as of the arc  $AM$ .

30. It is often convenient to use some other radius than 1; in such case, the functions of the arc, to the radius 1, may be reduced to corresponding functions, to the radius  $R$ .

Let  $AOM$  represent any angle,  $AM$  an arc described from  $O$  as a centre with the radius 1,  $PM$  its sine;  $A'M'$  an arc described from  $O$  as a centre, with any radius  $R$ , and  $P'M'$  its sine. Then, because  $OPM$  and  $OP'M'$  are similar triangles, we shall have,



$$OM : PM :: OM' : P'M', \text{ or, } 1 : PM :: R : P'M';$$

whence,

$$PM = \frac{P'M'}{R}, \quad \text{and,} \quad P'M' = PM \times R;$$

and similarly for each of the other functions.

That is, *any function of an arc whose radius is 1, is equal to the corresponding function of an arc whose radius is  $R$ , divided by that radius. Also, any function of an arc whose radius is  $R$ , is equal to the corresponding function of an arc whose radius is 1, multiplied by the radius  $R$ .*

By making these changes in any formula, the formula will be rendered *homogeneous*.

## TABLE OF NATURAL SINES.

31. A NATURAL SINE, COSINE, TANGENT, OR COTANGENT, is the sine, cosine, tangent, or cotangent of an arc whose radius is 1.

A TABLE OF NATURAL SINES is a table by means of which the natural sine, cosine, tangent, or cotangent of any arc, may be found.

Such a table might be used for all the purposes of trigonometrical computation, but it is found more convenient to employ a table of logarithmic sines, as explained in the next article.

## TABLE OF LOGARITHMIC SINES.

32. A LOGARITHMIC SINE, COSINE, TANGENT, OR COTANGENT is the logarithm of the sine, cosine, tangent, or cotangent of an arc whose radius is 10,000,000,000.

A TABLE OF LOGARITHMIC SINES is a table from which the logarithmic sine, cosine, tangent, or cotangent of any arc may be found.

The logarithm of the tabular radius is 10.

Any *logarithmic* function of an arc may be found by multiplying the corresponding *natural* function by 10,000,000,000 (Art. 30), and then taking the logarithm of the result; or more simply, by taking the logarithm of the corresponding *natural* function, and then adding 10 to the result (Art. 5).

33. In the table appended, the logarithmic functions are given for every *minute* from  $0^\circ$  up to  $90^\circ$ . In addition, their rates of change for each *second*, are given in the column headed "D."

The method of computing the numbers in the column headed "D," will be understood from a single example. The



logarithmic sines of  $27^{\circ} 34'$ , and of  $27^{\circ} 35'$ , are, respectively, 9.665375 and 9.665617. The difference between their mantissas is 242; this, divided by 60, the number of seconds in one minute, gives 4.03, which is the change in the mantissa for  $1''$ , between the limits  $27^{\circ} 34'$  and  $27^{\circ} 35'$ .

For the sine and cosine, there are separate columns of differences, which are written to the right of the respective columns; but for the tangent and cotangent, there is but a single column of differences, which is written between them. The logarithm of the tangent increases, just as fast as that of the cotangent decreases, and the reverse, their sum being always equal to 20. The reason of this is, that the product of the tangent and cotangent is always equal to the square of the radius; hence, the sum of their logarithms must always be equal to twice the logarithm of the radius, or 20.

The angle obtained by taking the degrees from the top of the page, and the minutes from any line on the left hand of the page, is the complement of that obtained by taking the degrees from the bottom of the page, and the minutes from the same line on the right hand of the page. But, by definition, the cosine and the cotangent of an arc are, respectively, the sine and the tangent of the complement of that arc (Arts. 26 and 28): hence, the columns designated *sine* and *tang*, at the top of the page, are designated *cosine* and *cotang* at the bottom.

#### USE OF THE TABLE.

*To find the logarithmic functions of an arc which is expressed in degrees and minutes.*

34. If the arc is less than  $45^{\circ}$ , look for the degrees at the top of the page, and for the minutes in the left hand column; then follow the corresponding horizontal line till you

come to the column designated at the top by *sine*, *cosine*, *tang*, or *cotang*, as the case may be; the number there found is the logarithm required. Thus,

$$\begin{array}{l} \log \sin 19^\circ 55' \quad . \quad . \quad . \quad 9.532312 \\ \log \tan 19^\circ 55' \quad . \quad . \quad . \quad 9.559097 \end{array}$$

If the angle is greater than  $45^\circ$ , look for the degrees at the bottom of the page, and for the minutes in the right hand column; then follow the corresponding horizontal line backwards till you come to the column designated at the bottom by *sine*, *cosine*, *tang*, or *cotang*, as the case may be; the number there found is the logarithm required. Thus,

$$\begin{array}{l} \log \cos 52^\circ 18' \quad . \quad . \quad . \quad 9.786416 \\ \log \tan 52^\circ 18' \quad . \quad . \quad . \quad 10.111884 \end{array}$$

*To find the logarithmic functions of an arc which is expressed in degrees, minutes, and seconds.*

35. Find the logarithm corresponding to the degrees and minutes as before; then multiply the corresponding number taken from the column headed "D," by the number of seconds, and add the product to the preceding result, for the sine or tangent, and subtract it therefrom for the cosine or cotangent.

#### EXAMPLES.

1. Find the logarithmic sine of  $40^\circ 26' 28''$ .

#### OPERATION.

$$\begin{array}{l} \log \sin 40^\circ 26' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 9.811952 \\ \text{Tabular difference} \quad 2.47 \\ \text{No. of seconds} \quad \quad 28 \\ \text{Product} \quad . \quad . \quad . \quad 69.16 \quad \text{to be added} \quad . \quad . \quad \underline{69} \\ \log \sin 40^\circ 26' 28'' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \underline{\underline{9.812021}} \end{array}$$

The same rule is followed for decimal parts, as in Art. 12.

2. Find the logarithmic cosine of  $53^{\circ} 40' 40''$ .

OPERATION.

log cos $53^{\circ} 40'$	. . . . .	9.772675
Tabular difference	2.86	
No. of seconds	<u>40</u>	
Product . . .	114.40	to be subtracted <u>114</u>
log cos $53^{\circ} 40' 40''$	. . . . .	<u>9.772561</u>

If the arc is greater than  $90^{\circ}$ , we find the required function of its supplement (Arts. 26 and 28).

3. Find the logarithmic tangent of  $118^{\circ} 18' 25''$ .

OPERATION.

	180°	
Given arc . . . . .	<u>118° 18' 25''</u>	
Supplement . . . . .	61° 41' 35''	
log tan $61^{\circ} 41'$	. . . . .	10.268556
Tabular difference	5.04	
No. of seconds	<u>35</u>	
Product . . .	176.40	to be added . <u>176</u>
log tan $118^{\circ} 18' 25''$	. . . . .	<u>10.268732</u>

4. Find the logarithmic sine of  $32^{\circ} 18' 35''$ .

*Ans.* 9.727945.

5. Find the logarithmic cosine of  $95^{\circ} 18' 24''$ .

*Ans.* 8.966080.

6. Find the logarithmic cotangent of  $126^{\circ} 23' 50''$ .

*Ans.* 10.132333.



*To find the arc corresponding to any logarithmic function.*

36. This is done by reversing the preceding rule :  
Look in the proper column of the table for the given logarithm ; if it is found there, the degrees are to be taken from the top or bottom, and the minutes from the left or right hand column, as the case may be. If the given logarithm is not found in the table, then find the next less logarithm, and take from the table the corresponding degrees and minutes, and set them aside. Subtract the logarithm found in the table, from the given logarithm, and divide the remainder by the corresponding tabular difference. The quotient will be seconds, which must be *added* to the degrees and minutes set aside, in the case of a sine or tangent, and *subtracted*, in the case of a cosine or a cotangent.

#### EXAMPLES.

1. Find the arc corresponding to the logarithmic sine 9.422248.

#### OPERATION.

Given logarithm	. . .	9.422248	
Next less in table	. . .	<u>9.421857</u>	. . . 15° 19'
Tabular difference	7.68 )	391.00	( 51'', to be added.

Hence, the required arc is 15° 19' 51''.

2. Find the arc corresponding to the logarithmic cosine 9.427485.

#### OPERATION.

Given logarithm	. . .	9.427485	
Next less in table	. . .	<u>9.427354</u>	. . . 74° 29'.
Tabular difference	7.58 )	131.00	( 17'', to be sub <del>t</del> .

Hence, the required arc is 74° 28' 43''.

3. Find the arc corresponding to the logarithmic sine 9.880054. *Ans.*  $49^{\circ} 20' 50''$ .

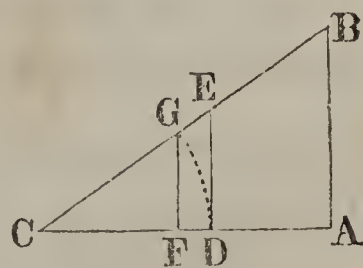
4. Find the arc corresponding to the logarithmic cotangent 10.008688. *Ans.*  $44^{\circ} 25' 37''$ .

5. Find the arc corresponding to the logarithmic cosine 9.944599. *Ans.*  $28^{\circ} 19' 45''$ .

### SOLUTION OF RIGHT-ANGLED TRIANGLES.

37. In what follows, we shall designate the three angles of every triangle, by the capital letters  $A$ ,  $B$ , and  $C$ ,  $A$  denoting the right angle; and the sides lying opposite the angles, by the corresponding small letters  $a$ ,  $b$ , and  $c$ . Since the order in which these letters are placed may be changed, it follows that whatever is proved with the letters placed in any given order, will be equally true when the letters are correspondingly placed in any other order.

Let  $CAB$  represent any triangle, right-angled at  $A$ . With  $C$  as a centre, and a radius  $CD$ , equal to 1, describe the arc  $DG$ , and draw  $GF$  and  $DE$  perpendicular to  $CA$ : then will  $FG$  be the sine of the angle  $C$ ,  $CF$  will be its cosine, and  $DE$  its tangent.



Since the three triangles  $CFG$ ,  $CDE$ , and  $CAB$  are similar (B. IV., P. XVIII.), we may write the proportions,

$$CB : CG :: AB : FG, \quad \text{or,} \quad a : 1 :: c : \sin C;$$

$$CB : CG :: CA : CF, \quad \text{or,} \quad a : 1 :: b : \cos C;$$

$$CA : CD :: AB : DE, \quad \text{or,} \quad b : 1 :: c : \tan C;$$

hence, we have (B. II., P. I.),

$$\left. \begin{aligned} c &= a \sin C \quad \cdot \quad \cdot \quad \cdot \quad (1.) \\ b &= a \cos C \quad \cdot \quad \cdot \quad \cdot \quad (2.) \\ c &= b \tan C \quad \cdot \quad \cdot \quad \cdot \quad (3.) \end{aligned} \right\} \therefore \left\{ \begin{aligned} \sin C &= \frac{c}{a}, \quad \cdot \quad \cdot \quad \cdot \quad (4.) \\ \cos C &= \frac{b}{a}, \quad \cdot \quad \cdot \quad \cdot \quad (5.) \\ \tan C &= \frac{c}{b}, \quad \cdot \quad \cdot \quad \cdot \quad (6.) \end{aligned} \right.$$

Translating these formulas into ordinary language, we have the following

#### PRINCIPLES.

1. *The perpendicular of any right-angled triangle is equal to the hypotenuse into the sine of the angle at the base.*

2. *The base is equal to the hypotenuse into the cosine of the angle at the base.*

3. *The perpendicular is equal to the base into the tangent of the angle at the base.*

4. *The sine of the angle at the base is equal to the perpendicular divided by the hypotenuse.*

5. *The cosine of the angle at the base is equal to the base divided by the hypotenuse.*

6. *The tangent of the angle at the base is equal to the perpendicular divided by the base.*

Either side about the right angle may be regarded as the base; in which case, the other is to be regarded as the perpendicular. We see, then, that the above principles are sufficient for the solution of every case of right-angled triangles. When the table of logarithmic sines is used, in the solution, Formulas (1) to (6) must be made homogeneous, by substituting for  $\sin C$ ,  $\cos C$ , and  $\tan C$ , respectively,





Applying logarithms to Formula (8), we have,

$$\log b = \log a + \log \cos C - 10;$$

$\log a$	(749)	. . . .	2.874481	
$\log \cos C$	(47° 03' 10'')	. . . .	<u>9.833354</u>	
$\log b$	. . . . .		<u>2.707835</u>	$\therefore b = 510.31$

*Ans.*  $B = 42^\circ 56' 50''$ ,  $b = 510.31$ , and  $c = 548.255$

2. Given  $a = 439$ , and  $B = 27^\circ 38' 50''$ , to find  $C$ ,  $b$ , and  $c$ .

OPERATION.

$$C = 90^\circ - 27^\circ 38' 50'' = 62^\circ 21' 10'';$$

$\log a$	(439)	. . . .	2.642465	
$\log \sin C$	(62° 21' 10'')	. . . .	<u>9.947346</u>	
$\log c$	. . . . .		2.589811	$\therefore c = 388.875;$

$\log a$	(439)	. . . .	2.642465	
$\log \cos C$	(62° 21' 10'')	. . . .	<u>9.666543</u>	
$\log b$	. . . . .		2.309008	$\therefore b = 203.708.$

*Ans.*  $C = 62^\circ 21' 10''$ ,  $b = 203.708$ , and  $c = 388.875$ .

3. Given  $a = 125.7$  yds., and  $B = 75^\circ 12'$ , to find the other parts.

*Ans.*  $C = 14^\circ 48'$ ,  $b = 121.53$  yds., and  $c = 32.11$  yds.

4. Given  $a = 325$  ft., and  $C = 27^\circ 34'$ , to find the other parts.

*Ans.*  $B = 62^\circ 26'$ ,  $c = 150.4$  ft., and  $b = 288.1$  ft.

CASE II.

*Given one of the sides about the right angle and one of the acute angles, to find the remaining parts.*

39. The other acute angle may be found by subtracting the given one from  $90^\circ$ .

The hypotenuse may be found by Formula (7), and the unknown side about the right angle, by Formula (8).

EXAMPLES.

1. Given  $c = 56.293$ , and  $C = 54^\circ 27' 39''$ , to find  $B$ ,  $a$ , and  $b$ .

OPERATION.

$$B = 90^\circ - 54^\circ 27' 39'' = 35^\circ 32' 21''.$$

Applying logarithms to Formula (7), we have,

$$\log c = \log a + \log \sin C - 10; \quad \text{whence,}$$

$$\log a = \log c + 10 - \log \sin C = \log c + (\text{a. c.}) \log \sin C;$$

log c	(56.293)	. . .	1.750454	
(a. c.) log sin C	(54° 27' 39'')	. .	0.089527	
log a	. . . . .	. .	<u>1.839981</u>	∴ a = 69.18.

Applying logarithms to Formula (8), we have,

$$\log b = \log a + \log \cos C - 10;$$

log a	(69.18)	. . . .	1.839981	
log cos C	(54° 27' 39'')	. .	<u>9.764370</u>	
log b	. . . . .	. .	<u>1.604351</u>	∴ b = 40.2114.

*Ans.*  $B = 35^\circ 32' 21''$ ,  $a = 69.18$ , and  $b = 40.2114$ .



2. Given  $c = 358$ , and  $B = 28^\circ 47'$ , to find  $C$ ,  $a$ , and  $b$ .

OPERATION.

$$C = 90^\circ - 28^\circ 47' = 61^\circ 13'.$$

We have, as before,

$$\log a = \log c + (\text{a. c.}) \log \sin C,$$

and,  $\log b = \log a + \log \cos C - 10;$

$\log c$	(358)	. . .	2.553883	
(a. c.) $\log \sin C$	(61° 13')	. . .	0.057274	
$\log a$	. . . . .		<u>2.611157</u>	$\therefore a = 408.466;$

$\log a$	(313.776)	. .	2.611157	
$\log \cos C$	(61° 13')	. .	<u>9.682595</u>	
$\log b$	. . . . .		<u>2.293752</u>	$\therefore b = 196.676.$

*Ans.*  $C = 61^\circ 13'$ ,  $a = 408.466$ , and  $b = 196.676$ .

3. Given  $b = 152.67$  yds., and  $C = 50^\circ 18' 32''$ , to find the other parts.

*Ans.*  $B = 39^\circ 41' 28''$ ,  $c = 183.95$ , and  $a = 239.05$ .

4. Given  $c = 379.628$ , and  $C = 39^\circ 26' 16''$ , to find  $B$ ,  $a$ , and  $b$ .

*Ans.*  $B = 50^\circ 33' 44''$ ,  $a = 597.613$ , and  $b = 461.55$ .

### CASE III.

*Given the two sides about the right angle, to find the remaining parts.*

40. The angle at the base may be found by Formula (12), and the solution may be completed as in Case II.

## EXAMPLES.

1. Given  $b = 26$ , and  $c = 15$ , to find  $C$ ,  $B$ , and  $a$ .

## OPERATION.

Applying logarithms to Formula (12), we have,

$$\log \tan C = \log c + 10 - \log b = \log c + (\text{a. c.}) \log b;$$

$\log c$	(15)	.	.	.	.	1.176091	
(a. c.) $\log b$	(26)	.	.	.	.	8.585027	
$\log \tan C$		.	.	.	.	<u>9.761118</u>	$\therefore C = 29^\circ 58' 54''$ ;

$$B = 90^\circ - C = 60^\circ 01' 06''.$$

As in Case II.,  $\log a = \log c + (\text{a. c.}) \log \sin C$ ;

$\log a$	.	.	(15)	.	.	1.176091	
(a. c.) $\log \sin C$	(29° 58' 54'')					<u>0.301271</u>	
$\log a$	.	.	.	.	.	<u>1.477362</u>	$\therefore a = 30.017$ .

*Ans.*  $C = 29^\circ 58' 54''$ ,  $B = 60^\circ 01' 06''$ , and  $a = 30.017$ .

2. Given  $b = 1052$  yds., and  $c = 347.21$  yds., to find  $B$ ,  $C$ , and  $a$ .

$$B = 71^\circ 44' 05'', \quad C = 18^\circ 15' 55'', \quad \text{and} \quad a = 1108.05 \text{ yds.}$$

3. Given  $b = 122.416$ , and  $c = 118.297$ , to find  $B$ ,  $C$ , and  $a$ .

$$B = 45^\circ 58' 50'', \quad C = 44^\circ 1' 10'', \quad \text{and} \quad a = 170.235.$$

4. Given  $b = 103$ , and  $c = 101$ , to find  $B$ ,  $C$ , and  $a$ .

$$B = 45^\circ 33' 42'', \quad C = 44^\circ 26' 18'', \quad \text{and} \quad a = 144.256.$$

## CASE IV.

*Given the hypotenuse and either side about the right angle,  
to find the remaining parts.*

41. The angle at the base may be found by one of Formulas (10) and (11), and the remaining side may then be found by one of Formulas (7) and (8).

## EXAMPLES.

1. Given  $a = 2391.76$ , and  $b = 385.7$ , to find  $B$ ,  $C$ , and  $c$ .

## OPERATION.

Applying logarithms to Formula (11), we have,

$$\log \cos C = \log b + 10 - \log a = \log b + (\text{a. c.}) \log a;$$

$\log b$	$(385.7)$	$\cdot \cdot \cdot$	$2.586250$	
$(\text{a. c.}) \log a$	$(2391.76)$	$\cdot \cdot$	$6.621282$	
$\log \cos C$	$\cdot \cdot \cdot$		<u><math>9.207532</math></u>	$\therefore C = 80^\circ 43' 11''$ ;

$$B = 90^\circ - 80^\circ 43' 11'' = 9^\circ 16' 49''.$$

From Formula (7), we have,

$$\log c = \log a + \log \sin C - 10;$$

$\log a$	$(2391.76)$	$\cdot$	$3.378718$	
$\log \sin C$	$(80^\circ 43' 11'')$		<u><math>9.994278</math></u>	
$\log c$	$\cdot \cdot \cdot \cdot \cdot$		<u><math>3.372996</math></u>	$\therefore c = 2360.45.$

*Ans.*  $B = 9^\circ 16' 49''$ ,  $C = 80^\circ 43' 11''$ , and  $c = 2360.45$ .



2. Given  $a = 127.174$  yds., and  $c = 125.7$  yds., to find  $B$ ,  $C$ , and  $b$ .

OPERATION.

From Formula (10), we have,

$$\log \sin C = \log c + 10 - \log a = \log c + (\text{a. c.}) \log a ;$$

$$\begin{array}{rcl} \log c \quad (125.7) & . & . & . & 2.099335 \\ (\text{a. c.}) \log a \quad (127.174) & . & . & . & \underline{7.895602} \\ \log \sin c & . & . & . & \underline{9.994937} \end{array} \therefore C = 81^{\circ} 16' 6'' ;$$

$$B = 90^{\circ} - 81^{\circ} 16' 6'' = 8^{\circ} 43' 54''.$$

From Formula (8), we have,

$$\log b = \log a + \log \cos C - 10 ;$$

$$\begin{array}{rcl} \log a \quad (127.174) & . & 2.104398 \\ \log \cos C \quad (81^{\circ} 16' 6'') & . & \underline{9.181292} \\ \log b & . & . & . & \underline{1.285690} \end{array} \therefore b = 19.3.$$

*Ans.*  $B = 8^{\circ} 43' 54''$ ,  $C = 81^{\circ} 16' 6''$ , and  $b = 19.3$  yds.

3. Given  $a = 100$ , and  $b = 60$ , to find  $B$ ,  $C$ , and  $c$ .

*Ans.*  $B = 36^{\circ} 52' 11''$ ,  $C = 53^{\circ} 7' 49''$ , and  $c = 80$ .

4. Given  $a = 19.209$ , and  $c = 15$ , to find  $B$ ,  $C$ , and  $b$ .

*Ans.*  $B = 38^{\circ} 39' 30''$ ,  $C = 51^{\circ} 20' 30''$ ,  $b = 12$ .

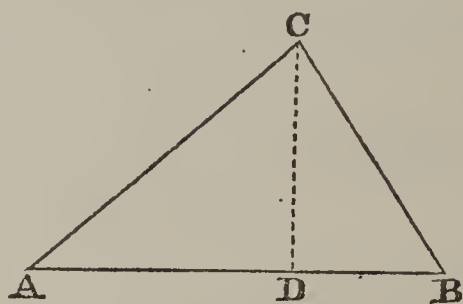
## SOLUTION OF OBLIQUE-ANGLED TRIANGLES.

42. In the solution of oblique-angled triangles, *four* cases may arise. We shall discuss these cases in order.

## CASE I.

*Given one side and two angles, to determine the remaining parts.*

43. Let  $ABC$  represent any oblique-angled triangle. From the vertex  $C$ , draw  $CD$  perpendicular to the base, forming two right-angled triangles  $ACD$  and  $BCD$ . Assume the notation of the figure.



From Formula (1), we have,

$$CD = b \sin A, \quad \text{and} \quad CD = a \sin B;$$

Equating these two values, we have,

$$b \sin A = a \sin B;$$

whence (B. II., P. II.),

$$a : b :: \sin A : \sin B. \quad . \quad . \quad (13.)$$

Since  $a$  and  $b$  are any two sides, and  $A$  and  $B$  the angles lying opposite to them, we have the following principle :

*The sides of a plane triangle are proportional to the sines of the opposite angles.*

It is to be observed that Formula (13) is true for any value of the radius. Hence, to solve a triangle, when a side and two angles are given :

First find the third angle, by subtracting the sum of the given angles from  $180^\circ$ ; then find each of the required sides by means of the principle just demonstrated.

EXAMPLES.

1. Given  $B = 58^\circ 07'$ ,  $C = 22^\circ 37'$ , and  $a = 408$ , to find  $A$ ,  $b$ , and  $c$ .

OPERATION.

$$\begin{array}{rcl} B & . & . & . & . & . & . & 58^\circ 07' \\ C & . & . & . & . & . & . & 22^\circ 37' \\ A & . & . & . & 180^\circ - & 80^\circ 44' & = & 99^\circ 16' \end{array}$$

To find  $b$ , write the proportion,

$$\sin A : \sin B :: a : b ;$$

that is, *the sine of the angle opposite the given side, is to the sine of the angle opposite the required side, as the given side is to the required side.*

Applying logarithms, and reducing, we have,

$$\log b = \log a + \log \sin B + (\text{a. c.}) \log \sin A - 10 ;$$

$$\begin{array}{rcl} \log a & . & . & (408) & . & . & . & 2.610660 \\ \log \sin B & (58^\circ 07') & . & . & . & . & . & 9.928972 \\ (\text{a. c.}) \log \sin A & (99^\circ 16') & . & . & . & . & . & 0.005705 \\ \log b & . & . & . & . & . & . & 2.545337 \quad \therefore b = 351.024. \end{array}$$

In like manner,

$$\log c = \log a + \log \sin C + (\text{a. c.}) \log \sin A - 10 ;$$

$$\begin{array}{rcl} \log a & . & . & (408) & . & . & . & 2.610660 \\ \log \sin C & (22^\circ 37') & . & . & . & . & . & 9.584968 \\ (\text{a. c.}) \log \sin A & (99^\circ 16') & . & . & . & . & . & 0.005705 \\ \log c & . & . & . & . & . & . & 2.201333 \quad \therefore c = 158.976. \end{array}$$

$$\text{Ans. } A = 99^\circ 16', \quad b = 351.024, \quad \text{and} \quad c = 158.976.$$



2. Given  $A = 38^\circ 25'$ ,  $B = 57^\circ 42'$ , and  $c = 400$ , to find  $C$ ,  $a$ , and  $b$ .

*Ans.*  $C = 83^\circ 53'$ ,  $a = 249.974$ ,  $b = 340.04$ .

3. Given  $A = 15^\circ 19' 51''$ ,  $C = 72^\circ 44' 05''$ , and  $c = 250.4$  yds, to find  $B$ ,  $a$ , and  $b$ .

*Ans.*  $B = 91^\circ 56' 04''$ ,  $a = 69.328$  yds.,  $b = 262.066$  yds.

4. Given  $B = 51^\circ 15' 35''$ ,  $C = 37^\circ 21' 25''$ , and  $a = 305.296$  ft., to find  $A$ ,  $b$ , and  $c$ .

*Ans.*  $A = 91^\circ 23'$ ,  $b = 238.1978$  ft.,  $c = 185.3$  ft.

#### CASE II.

*Given two sides and an angle opposite one of them, to find the remaining parts.*

44. The solution, in this case, is commenced by finding a second angle by means of Formula (13), after which we may proceed as in CASE I.; or, the solution may be completed by a continued application of Formula (13).

#### EXAMPLES.

1. Given  $A = 22^\circ 37'$ ,  $b = 216$ , and  $a = 117$ , to find  $B$ ,  $C$ , and  $c$ .

From Formula (13), we have,

$$a : b :: \sin A : \sin B ;$$

that is, *the side opposite the given angle, is to the side opposite the required angle, as the sine of the given angle is to the sine of the required angle.*

Whence, by the application of logarithms,

$$\log \sin B = \log b + \log \sin A + (\text{a. c.}) \log a - 10 ;$$

$$\log b \quad . \quad . \quad (216) \quad . \quad . \quad 2.334454$$

$$\log \sin A \quad (22^\circ 37') \quad . \quad . \quad 9.584968$$

$$(\text{a. c.}) \log a \quad . \quad . \quad (117) \quad . \quad . \quad 7.931814$$

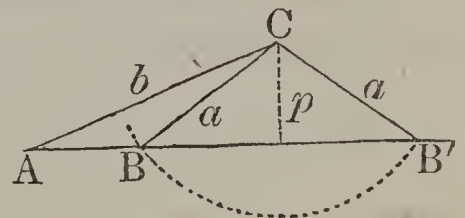
$$\log \sin B \quad . \quad . \quad . \quad 9.851236 \quad \therefore B = 45^\circ 13' 55'',$$

$$\text{and } B' = 134^\circ 46' 05''.$$

Hence, we find two values of  $B$ , which are supplements of each other, because the sine of any angle is equal to the sine of its supplement. This would seem to indicate that the problem admits of two solutions. It now remains to determine under what conditions there will be *two solutions*, *one solution*, or *no solution*.

There may be two cases: the given angle may be *acute*, or it may be *obtuse*.

*First Case.* Let  $ABC$  represent the triangle, in which the angle  $A$ , and the sides  $a$  and  $b$  are given. From  $C$  let fall



a perpendicular upon  $AB$ , prolonged if necessary, and denote its length by  $p$ . We shall have, from Formula (1), Art. 37,

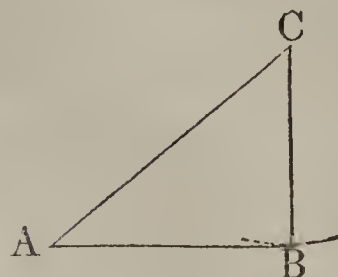
$$p = b \sin A ;$$

from which the value of  $p$  may be computed.

If  $a$  is intermediate in value between  $p$  and  $b$ , there will be *two solutions*. For, if with  $C$  as a centre, and  $a$  as a radius, an arc be described, it will cut the line  $AB$  in two points,  $B$  and  $B'$ , each of which being joined with  $C$ , will give a triangle which will conform to the conditions of the problem.

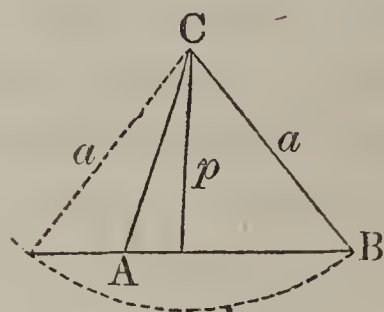
In this case, the angles  $B'$  and  $B$ , of the two triangles  $AB'C$  and  $ABC$ , will be supplements of each other.

If  $a = p$ , there will be but *one solution*. For, in this case, the arc will be tangent to  $AB$ , the two points  $B$  and  $B'$  will unite, and there will be but a single triangle formed.



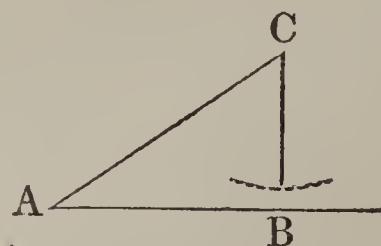
In this case, the angle  $ABC$  will be equal to  $90^\circ$ .

If  $a$  is greater than both  $p$  and  $b$ , there will also be but one solution. For, although the arc cuts  $AB$  in two points, and consequently gives two triangles, only one of them conforms to the conditions of the problem.

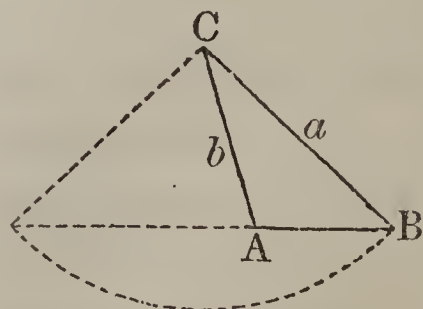


In this case, the angle  $ABC$  will be less than  $A$ , and consequently acute.

If  $a < p$ , there will be *no solution*. For, the arc can neither cut  $AB$ , nor be tangent to it.



*Second Case.* When the given angle  $A$  is obtuse, the angle  $ABC$  will be acute; the side  $a$  will be greater than  $b$ , and there will be but *one solution*.



In the example under consideration, there are two solutions, the first corresponding to  $B = 45^\circ 13' 55''$ , and the second to  $B' = 134^\circ 46' 05''$ .



In the first case, we have,

$$\begin{array}{l} A \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot 22^{\circ} 37' \\ B \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot 45^{\circ} 13' 55'' \\ C \cdot \cdot \cdot \cdot \cdot \cdot 180^{\circ} - \underline{67^{\circ} 50' 55''} = 112^{\circ} 09' 05''. \end{array}$$

As in Case I., we have,

$$\log c = \log b + \log \sin C + (\text{a. c.}) \log \sin B - 10 ;$$

$$\begin{array}{l} \log b \cdot \cdot (216) \cdot \cdot \cdot 2.334454 \\ \log \sin C (112^{\circ} 09' 05'') \cdot 9.966700 \\ (\text{a. c.}) \log \sin B (45^{\circ} 13' 55'') \cdot \underline{0.148764} \\ \log c \cdot \cdot \cdot \cdot \cdot \cdot \underline{2.449918} \therefore c = 281.785. \end{array}$$

$$\text{Ans. } B = 45^{\circ} 13' 55'', \quad C = 112^{\circ} 09' 05'', \quad \text{and } c = 281.785.$$

In the second case, we have,

$$\begin{array}{l} A \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot 22^{\circ} 37' \\ B' \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot 134^{\circ} 46' 05'' \\ C \cdot \cdot \cdot \cdot \cdot \cdot 180^{\circ} - \underline{157^{\circ} 23' 05''} = 22^{\circ} 36' 55'' ; \end{array}$$

and as before,

$$\begin{array}{l} \log b \cdot \cdot (216) \cdot \cdot \cdot 2.334454 \\ \log \sin C (22^{\circ} 36' 55'') \cdot 9.584943 \\ (\text{a. c.}) \log \sin B (134^{\circ} 46' 05'') \cdot \underline{0.148764} \\ \log c \cdot \cdot \cdot \cdot \cdot \cdot \underline{2.068161} \therefore c = 116.993. \end{array}$$

$$\text{Ans. } B' = 134^{\circ} 46' 05'', \quad C = 22^{\circ} 36' 55'', \quad \text{and } c = 116.993.$$

2. Given  $A = 32^{\circ}$ ,  $a = 40$ , and  $b = 50$ , to find  $B$ ,  $C$ , and  $c$ .

$$\text{Ans. } \left\{ \begin{array}{l} B = 41^{\circ} 28' 59'', \quad C = 106^{\circ} 31' 01'', \quad c = 72.368. \\ B = 138^{\circ} 31' 01'', \quad C = 9^{\circ} 28' 59'', \quad c = 12.436. \end{array} \right.$$

3. Given  $A = 18^\circ 52' 13''$ ,  $a = 27.465$  yds., and  $b = 13.189$  yds., to find  $B$ ,  $C$ , and  $c$ .

*Ans.*  $B = 8^\circ 56' 05''$ ,  $C = 152^\circ 11' 42''$ ,  $c = 39.611$  yds.

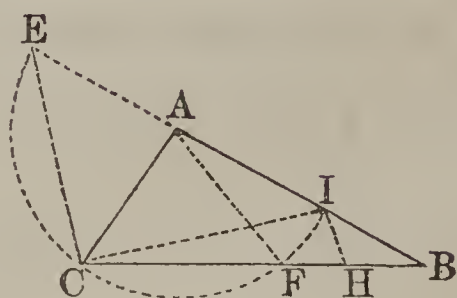
4. Given  $A = 32^\circ 15' 26''$ ,  $b = 176.21$  ft., and  $a = 94.047$  ft., to find  $B$ ,  $C$ , and  $c$ .

*Ans.*  $B = 90^\circ$ ,  $C = 57^\circ 44' 34''$ ,  $c = 149.014$  ft.

### CASE III.

*Given two sides and their included angle, to find the remaining parts.*

45. Let  $ABC$  represent any plane triangle,  $AB$  and  $AC$  any two sides, and  $A$  their included angle. With  $A$  as a centre, and  $AC$ , the shorter of the two sides, as a radius, describe a semi-circle meeting  $AB$  in  $I$ , and the prolongation of  $AB$  in  $E$ . Draw  $CI$  and  $EC$ , and through  $I$  draw  $IH$  parallel to  $EC$ .



Because  $ECI$  is an angle inscribed in a semicircle, it is a right angle (B. III. P. XVIII., C. 2); and consequently, both  $CE$  and  $IH$  are perpendicular to  $CI$ . The angle  $EAC$  being external to the triangle  $ABC$ , is equal to the sum of the opposite interior angles, that is, equal to  $C$  plus  $B$ ; the angle  $EAC$  being also external to the isosceles triangle  $AIC$ , it is equal to twice the angle  $AIC$ : hence, twice the angle  $AIC$  is equal to  $C$  plus  $B$ , or,

$$AIC = \frac{1}{2}(C + B).$$

The angle  $ICB$  is equal to  $AIC$  diminished by the angle  $IBC$  (B. I., P. XXV., C. 6); that is,

$$ICH = \frac{1}{2}(C + B) - B = \frac{1}{2}(C - B).$$

From the two right-angled triangles  $ICE$  and  $ICH$ , we have (Formula 3, Art. 37),

$$EC = IC \tan \frac{1}{2}(C + B), \quad \text{and} \quad IH = IC \tan \frac{1}{2}(C - B);$$

hence, from the preceding equations, we have, after omitting the equal factor  $IC$  (B. II., P. VII.),

$$EC : IH :: \tan \frac{1}{2}(C + B) : \tan \frac{1}{2}(C - B).$$

The triangles  $ECB$  and  $IHB$  being similar, their homologous sides are proportional; and because  $EB$  is equal to  $AB + AC$ , and  $IB$  to  $AB - AC$ , we shall have the proportion,

$$EC : IH :: AB + AC : AB - AC.$$

Combining the preceding proportions, and substituting for  $AB$  and  $AC$  their representatives  $c$  and  $b$ , we have,

$$c + b : c - b :: \tan \frac{1}{2}(C + B) : \tan \frac{1}{2}(C - B) \quad . \quad . \quad (14.)$$

Hence, we have the following principle:

*In any plane triangle, the sum of the sides including either angle, is to their difference, as the tangent of half the sum of the two other angles, is to the tangent of half their difference.*

The half sum of the angles may be found by subtracting the given angle from  $180^\circ$ , and dividing the remainder by 2; the half difference may be found by means of the principle just demonstrated. Knowing the half sum and the half



difference, the greater angle is found by adding the half difference to the half sum, and the less angle is found by subtracting the half difference from the half sum. Then the solution is completed as in Case I.

## EXAMPLES.

1. Given  $c = 540$ ,  $b = 450$ , and  $A = 80^\circ$ , to find  $B$ ,  $C$ , and  $a$ .

## OPERATION.

$$c + b = 990 ; \quad c - b = 90 ; \quad \frac{1}{2}(C + B) = \frac{1}{2}(180^\circ - 80^\circ) = 50^\circ.$$

Applying logarithms to Formula (14), we have,

$$\log \sin \frac{1}{2}(C - B) = \log (c - b) + \log \tan \frac{1}{2}(C + B) + (\text{a. c.}) \log (c + b) - 10 ;$$

$\log (c - b)$	$\cdot \cdot \cdot (90)$	1.954243	
$\log \tan \frac{1}{2}(C + B)$	$(50^\circ)$	10.076187	
$(\text{a. c.}) \log (c + b)$	$\cdot \cdot \cdot (990)$	7.004365	
		<hr/>	
$\log \tan \frac{1}{2}(C - B)$		9.034795	$\therefore \frac{1}{2}(C - B) = 6^\circ 11'$

$$C = 50^\circ + 6^\circ 11' = 56^\circ 11' ; \quad B = 50^\circ - 6^\circ 11' = 43^\circ 49'.$$

From Formula (13), we have,

$$\log a = \log c + \log \sin A + (\text{a. c.}) \log \sin C - 10 ;$$

$\log c$	$\cdot \cdot \cdot (540)$	$\cdot \cdot \cdot$	2.732394	
$\log \sin A$	$(80^\circ)$	$\cdot \cdot \cdot$	9.993351	
$(\text{a. c.}) \log \sin C$	$(56^\circ 11')$	$\cdot$	0.080492	
			<hr/>	
$\log a$	$\cdot \cdot \cdot \cdot \cdot$		2.806237	$\therefore a = 640.082.$

$$\text{Ans. } B = 43^\circ 49', \quad C = 56^\circ 11', \quad a = 640.082.$$

2. Given  $c = 1686$  yds.,  $b = 960$  yds., and  $A = 128^\circ 04'$ , to find  $B$ ,  $C$ , and  $a$ .

*Ans.*  $B = 18^\circ 21' 21''$ ,  $C = 33^\circ 34' 39''$ ,  $a = 2400$  yds.

3. Given  $a = 18.739$  yds.,  $b = 7.642$  yds., and  $C = 45^\circ 18' 28''$ , to find  $A$ ,  $B$ , and  $c$ .

*Ans.*  $A = 112^\circ 34' 13''$ ,  $B = 22^\circ 07' 19''$ ,  $c = 14.426$  yds.

4. Given  $a = 464.7$  yds.,  $b = 289.3$  yds., and  $C = 87^\circ 03' 48''$ , to find  $A$ ,  $B$ , and  $c$ .

*Ans.*  $A = 60^\circ 13' 39''$ ,  $B = 32^\circ 42' 33''$ ,  $c = 534.66$  yds.

5. Given  $a = 16.9584$  ft.,  $b = 11.9613$  ft., and  $C = 60^\circ 43' 36''$ , to find  $A$ ,  $B$ , and  $c$ .

*Ans.*  $A = 76^\circ 04' 10''$ ,  $B = 43^\circ 12' 14''$ ,  $c = 15.22$  ft.

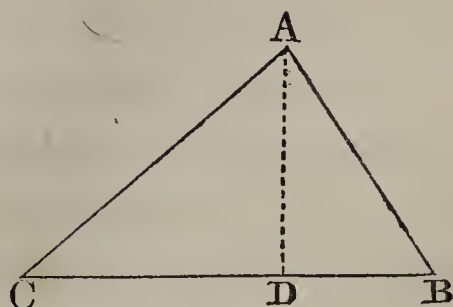
6. Given  $a = 3754$ ,  $b = 3277.628$ , and  $C = 57^\circ 53' 17''$ , to find  $A$ ,  $B$ , and  $c$ .

*Ans.*  $A = 68^\circ 02' 25''$ ,  $B = 54^\circ 04' 18''$ ,  $c = 3428.512$ .

#### CASE IV.

*Given the three sides of a triangle, to find the remaining parts.\**

46. Let  $ABC$  represent any plane triangle, of which  $BC$  is the longest side. Draw  $AD$  perpendicular to the base, dividing it into two segments  $BD$  and  $CD$ .




---

\* The angles may be found by Formula (A) or (B), Lemma. Pages 109, and 110, Mensuration.

From the right-angled triangles  $BAD$  and  $CAD$ , we have,

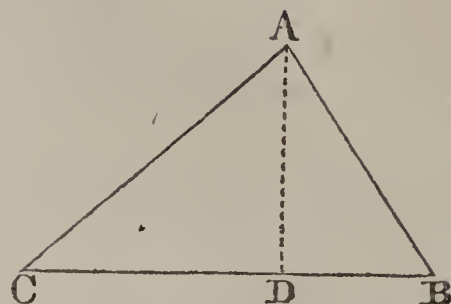
$$\overline{AD}^2 = \overline{AB}^2 - \overline{BD}^2, \quad \text{and} \quad \overline{AD}^2 = \overline{AC}^2 - \overline{DC}^2;$$

Equating these values of  $\overline{AD}^2$ , we have,

$$\overline{AB}^2 - \overline{BD}^2 = \overline{AC}^2 - \overline{DC}^2;$$

whence, by transposition,

$$\overline{AC}^2 - \overline{AB}^2 = \overline{DC}^2 - \overline{BD}^2.$$



Factoring each member, we have,

$$(AC + AB)(AC - AB) = (DC + BD)(DC - BD).$$

Converting this equation into a proportion (B. II., P. II.), we have,

$$DC + BD : AC + AB :: AC - AB : DC - BD;$$

or, denoting the segments by  $s$  and  $s'$ , and the sides of the triangle by  $a$ ,  $b$ , and  $c$ ,

$$s + s' : b + c :: b - c : s - s'; \quad (15.)$$

that is, if in any plane triangle, a line be drawn from the vertex of the vertical angle perpendicular to the base, dividing it into two segments; then,

*The sum of the two segments, or the whole base, is to the sum of the two other sides, as the difference of these sides is to the difference of the segments.*

The half difference added to the half sum, gives the greater, and the half difference subtracted from the half sum gives the less segment. We shall then have two right-angled triangles, in each of which we know the hypotenuse and the base; hence, the angles of these triangles may be found, and consequently, those of the given triangle.



EXAMPLES.

1. Given  $a = 40$ ,  $b = 34$ , and  $c = 25$ , to find  $A$ ,  $B$ , and  $C$ .

OPERATION.

Applying logarithms to Formula (15), we have,

$$\log (s - s') = \log (b + c) + \log (b - c) + (\text{a. c.}) \log (s + s') - 10;$$

$$\log (b + c) \cdot (59) \cdot \cdot \cdot 1.770852$$

$$\log (b - c) \cdot (9) \cdot \cdot \cdot 0.954243$$

$$(\text{a. c.}) \log (s + s') \cdot (40) \cdot \cdot \cdot 8.397940$$

$$\log (s - s') \cdot \cdot \cdot \cdot \underline{1.123035} \quad \therefore s - s' = 13.275.$$

$$s = \frac{1}{2}(s + s') + \frac{1}{2}(s - s') = 26.6375$$

$$s' = \frac{1}{2}(s + s') - \frac{1}{2}(s - s') = 13.3625$$

From Formula (11), we find,

$$\log \cos C = \log s + (\text{a. c.}) \log b \quad \therefore C = 38^\circ 25' 20'', \quad \text{and}$$

$$\log \cos B = \log s' + (\text{a. c.}) \log c \quad \therefore B = 57^\circ 41' 25''$$

$$\underline{96^\circ 06' 45''}$$

$$A = 180^\circ - 96^\circ 06' 45'' = 83^\circ 53' 15''.$$

2. Given  $a = 6$ ,  $b = 5$ , and  $c = 4$ , to find  $A$ ,  $B$ , and  $C$ .

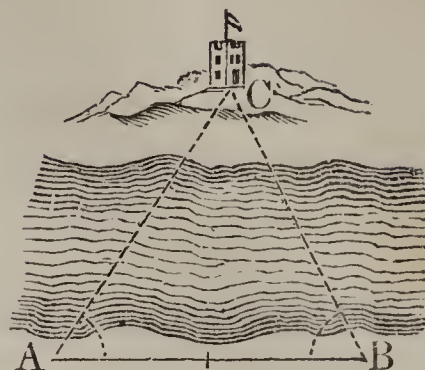
$$\text{Ans. } A = 82^\circ 49' 09'', \quad B = 55^\circ 46' 16'', \quad C = 41^\circ 24' 35''.$$

3. Given  $a = 71.2$  yds.,  $b = 64.8$  yds., and  $c = 37.4$  yds., to find  $A$ ,  $B$ , and  $C$ .

$$\text{Ans. } A = 83^\circ 44' 32'', \quad B = 64^\circ 46' 56'', \quad C = 31^\circ 28' 30''.$$

## PROBLEMS.

1. Knowing the distance  $AB$ , equal to 600 yards, and the angles  $BAC = 57^\circ 35'$ ,  $ABC = 64^\circ 51'$ , find the two distances  $AC$  and  $BC$ .

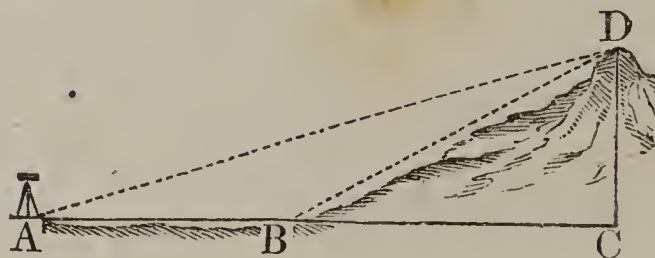


*Ans.*  $AC = 643.49$  yds.,  $BC = 600.11$  yds.

2. At what horizontal distance from a column, 200 feet high, will it subtend an angle of  $31^\circ 17' 12''$ ?

*Ans.* 329.114 ft.

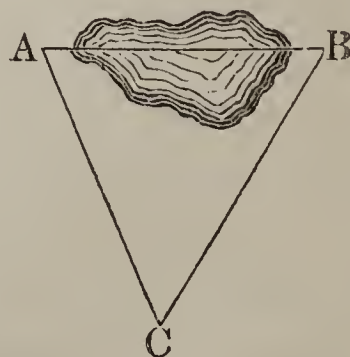
3. Required the height of a hill  $D$  above a horizontal plane  $AB$ , the distance between  $A$  and  $B$  being equal to 975 yards,



and the angles of elevation at  $A$  and  $B$  being respectively  $15^\circ 36'$  and  $27^\circ 29'$ .

*Ans.*  $DC = 587.61$  yds.

4. The distances  $AC$  and  $BC$  are found by measurement to be, respectively, 588 feet and 672 feet, and their included angle  $55^\circ 40'$ . Required the distance  $AB$ .



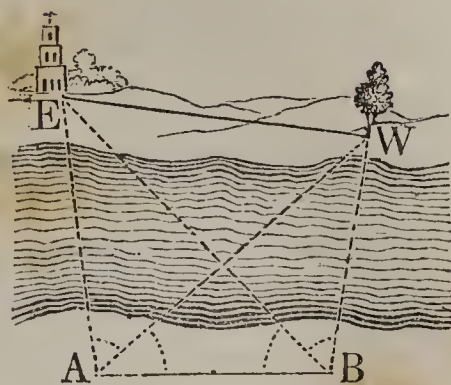
*Ans.* 592.967 ft.

5. Being on a horizontal plane, and wanting to ascertain the height of a tower, standing on the top of an inaccessible hill, there were measured, the angle of elevation of the top of the hill  $40^\circ$ , and of the top of the tower  $51^\circ$ ; then measuring in a direct line 180 feet farther from the hill, the

angle of elevation of the top of the tower was  $33^\circ 45'$ ; required the height of the tower. *Ans.* 83.998 ft.

6. Wanting to know the horizontal distance between two inaccessible objects  $E$  and  $W$ , the following measurements were made :

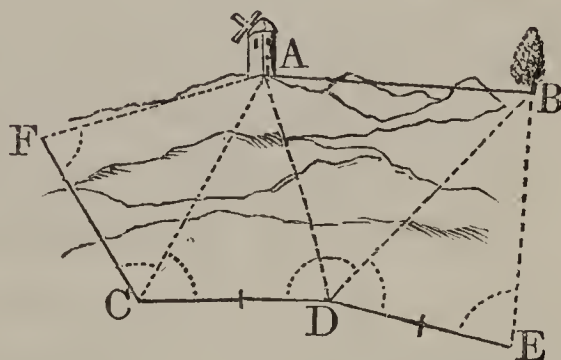
$$\text{viz : } \left\{ \begin{array}{l} AB = 536 \text{ yards} \\ BAW = 40^\circ 16' \\ WAE = 57^\circ 40' \\ ABE = 42^\circ 22' \\ EBW = 71^\circ 07'. \end{array} \right.$$



Required the distance  $EW$ .

*Ans.* 939.634 yds.

7. Wanting to know the horizontal distance between two inaccessible objects  $A$  and  $B$ , and not finding any station from which both of them could be seen, two points  $C$  and  $D$ , were chosen at a distance from each other



equal to 200 yards; from the former of these points,  $A$  could be seen, and from the latter,  $B$ ; and at each of the points  $C$  and  $D$ , a staff was set up. From  $C$ , a distance  $CF$  was measured, not in the direction  $DC$ , equal to 200 yards, and from  $D$ , a distance  $DE$ , equal to 200 yards, and the following angles taken :

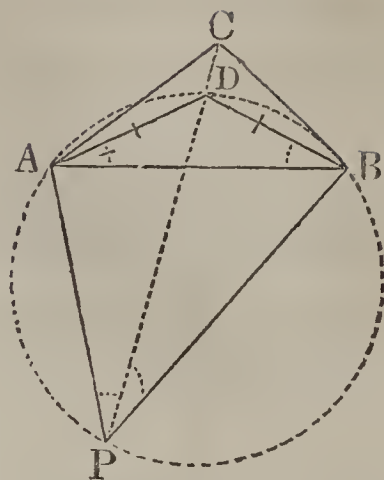
$$\begin{array}{lll} AFC = 83^\circ 00', & BDE = 54^\circ 30', & ACD = 53^\circ 30', \\ BDC = 156^\circ 25', & ACF = 54^\circ 31', & BED = 88^\circ 30'. \end{array}$$

Required the distance  $AB$ .

*Ans.* 345 467 yds.



8. The distances  $AB$ ,  $AC$ , and  $BC$ , between the points  $A$ ,  $B$ , and  $C$ , are known; viz. :  $AB = 800$  yds.,  $AC = 600$  yds., and  $BC = 400$  yds. From a fourth point  $P$ , the angles  $APC$  and  $BPC$  are measured; viz. :  $APC = 33^\circ 45'$ , and  $BPC = 22^\circ 30'$ .



Required the distances  $AP$ ,  $BP$ , and  $CP$ .

$$\text{Ans. } \begin{cases} AP = 710.193 \text{ yds.} \\ BP = 934.291 \text{ yds.} \\ CP = 1042.522 \text{ yds.} \end{cases}$$

This problem is used in locating the position of buoys in maritime surveying, as follows. Three points  $A$ ,  $B$ , and  $C$ , on shore are known in position. The surveyor stationed at a buoy  $P$ , measures the angles  $APC$  and  $BPC$ . The distances  $AP$ ,  $BP$ , and  $CP$ , are then found as follows :

Suppose the circumference of a circle to be described through the points  $A$ ,  $B$ , and  $P$ . Draw  $CP$ , cutting the circumference in  $D$ , and draw the lines  $DB$  and  $DA$ .

The angles  $CPB$  and  $DAB$ , being inscribed in the same segment, are equal (B. III., P. XVIII., C. 1); for a like reason, the angles  $CPA$  and  $DBA$  are equal: hence, in the triangle  $ADB$ , we know two angles and one side; we may, therefore, find the side  $DB$ . In the triangle  $ACB$ , we know the three sides, and we may compute the angle  $B$ . Subtracting from this the angle  $DBA$ , we have the angle  $DBC$ . Now, in the triangle  $DBC$ , we have two sides and their included angle, and we can find the angle  $DCB$ . Finally, in the triangle  $CPB$ , we have two angles and one side, from which data we can find  $CP$  and  $BP$ . In like manner, we can find  $AP$ .

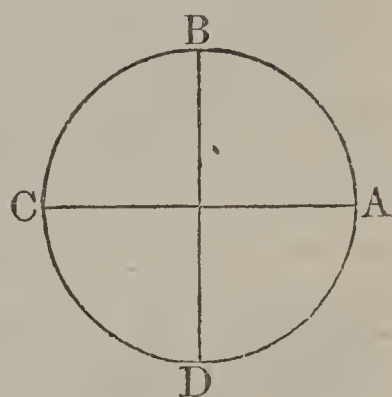
# ANALYTICAL TRIGONOMETRY.

---

47. ANALYTICAL TRIGONOMETRY is that branch of Mathematics which treats of the general properties and relations of trigonometrical functions.

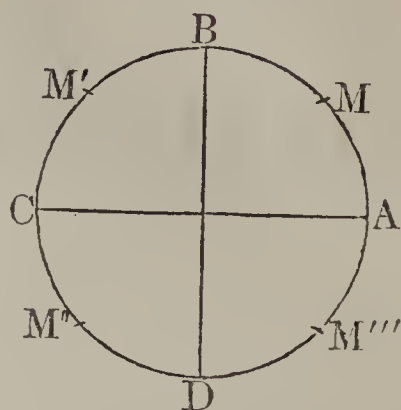
## DEFINITIONS AND GENERAL PRINCIPLES.

48. Let  $ABCD$  represent a circle whose radius is 1, and suppose its circumference to be divided into four equal parts, by the diameters  $AC$  and  $BD$ , drawn perpendicular to each other. The horizontal diameter  $AC$ , is called the *initial diameter*; the vertical diameter  $BD$ , is called the *secondary diameter*; the point  $A$ , from which arcs are usually reckoned, is called the *origin of arcs*, and the point  $B$ ,  $90^\circ$  distant, is called the *secondary origin*. Arcs estimated from  $A$ , around towards  $B$ , that is, in a direction contrary to that of the motion of the hands of a watch, are considered *positive*; consequently, those reckoned in a contrary direction must be regarded as *negative*.



The arc  $AB$ , is called the *first quadrant*; the arc  $BC$ , the *second quadrant*; the arc  $CD$ , the *third quadrant*; and the arc  $CA$ , the *fourth quadrant*. The point at which

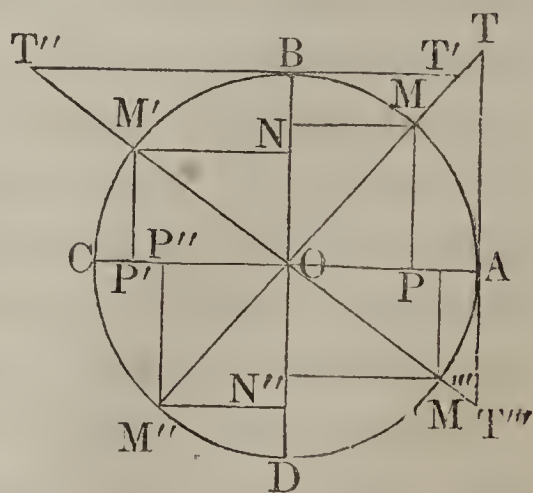
an arcs terminates, is called its *extremity*, and an arc is said to be in that quadrant in which its extremity is situated. Thus, the arc  $AM$  is in the *first quadrant*, the arc  $AM'$  in the *second*, the arc  $AM''$  in the *third*, and the arc  $AM'''$  in the *fourth*.



49. The *complement* of an arc has been defined to be the difference between that arc and  $90^\circ$  (Art. 23); geometrically considered, the *complement* of an arc is the arc included between the extremity of the arc and the secondary origin. Thus,  $MB$  is the complement of  $AM$ ;  $M'B$ , the complement of  $AM'$ ;  $M''B$ , the complement of  $AM''$ , and so on. When the arc is greater than a quadrant, the complement is negative, according to the conventional principle agreed upon (Art. 48).

The *supplement* of an arc has been defined to be the difference between that arc and  $180^\circ$  (Art. 24); geometrically considered, it is the arc included between the extremity of the arc and the left hand extremity of the initial diameter. Thus,  $MC$  is the supplement of  $AM$ , and  $M''C$  the supplement of  $AM''$ . The supplement is negative, when the arc is greater than two quadrants.

50. The *sine* of an arc is the distance from the initial diameter to the extremity of the arc. Thus,  $PM$  is the sine of  $AM$ , and  $P''M''$  is the sine of the arc  $AM''$ . The term *distance*, is used in the sense of *shortest* or *perpendicular distance*.





51. *The cosine of an arc is the distance from the secondary diameter to the extremity of the arc : thus,  $NM$  is the cosine of  $AM$ , and  $NM'$  is the cosine of  $AM'$ .*

The cosine may be measured on the initial diameter : thus,  $OP$  is equal to the cosine of  $AM$ , and  $OP'$  to the cosine of  $AM'$ .

52. *The versed-sine of an arc is the distance from the sine to the origin of arcs : thus,  $PA$  is the versed-sine of  $AM$ , and  $P'A$  is the versed-sine of  $AM'$ .*

53. *The co-versed-sine of an arc is the distance from the cosine to the secondary origin : thus,  $NB$  is the co-versed-sine of  $AM$ , and  $N''B$  is the co-versed-sine of  $AM''$ .*

54. *The tangent of an arc is that part of a perpendicular to the initial diameter, at the origin of arcs, included between the origin and the prolongation of the diameter through the extremity of the arc : thus,  $AT$  is the tangent of  $AM$ , or of  $AM''$ , and  $AT''$  is the tangent of  $AM'$ , or of  $AM'''$ .*

55. *The cotangent of an arc is that part of a perpendicular to the secondary diameter, at the secondary origin, included between the secondary origin and the prolongation of the diameter through the extremity of the arc : thus,  $BT'$  is the cotangent of  $AM$ , or of  $AM''$ , and  $BT''$  is the cotangent of  $AM'$ , or of  $AM'''$ .*

56. *The secant of an arc is the distance from the centre of the arc to the extremity of the tangent : thus,  $OT$  is the secant of  $AM$ , or of  $AM''$ , and  $OT'''$  is the secant of  $AM'$ , or of  $AM'''$ .*

57. *The cosecant of an arc is the distance from the*

centre of the arc to the extremity of the cotangent : thus,  $OT'$  is the cosecant of  $AM$ , or of  $AM''$ , and  $OT'''$  is the cosecant of  $AM'$ , or of  $AM'''$ .

The term *co*, in combination, is equivalent to *complement of*; thus, the *cosine* of an arc is the same as the *sine* of the *complement* of that arc, the *cotangent* is the same as the *tangent* of the *complement*, and so on.

The eight *trigonometrical functions* above defined are also called *circular functions*.

#### RULES FOR DETERMINING THE ALGEBRAIC SIGNS OF CIRCULAR FUNCTIONS.

58. All distances estimated upwards are regarded as positive; consequently, all distances estimated downwards must be considered negative.

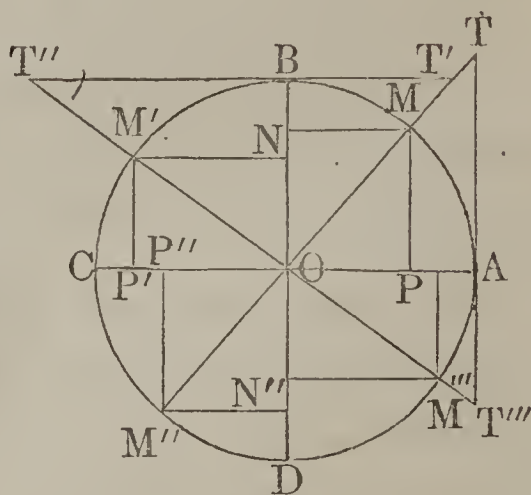
Thus,  $AT$ ,  $PM$ ,  $NB$ ,  $P'M'$ , are positive, and  $AT'''$ ,  $P'''M'''$ ,  $P''M''$ , &c., are negative.

All distances estimated towards the right are regarded as positive; consequently, all distances estimated towards the left must be considered negative.

Thus,  $NM$ ,  $BT'$ ,  $PA$ , &c., are positive, and  $N'M'$ ,  $BT''$ , &c., are negative.

All distances estimated from the centre in a direction towards the extremity of the arc are regarded as positive; consequently, all distances estimated in a direction from the second extremity of the arc must be considered negative.

Thus,  $OT$ , regarded as the secant of  $AM$ , is estimated in a direction towards  $M$ , and is positive; but  $OT$ , re-



garded as the secant of  $AM''$ , is estimated in a direction from  $M''$ , and is negative.

These conventional rules, enable us at once to give the proper sign to any function of an arc in any quadrant.

59. In accordance with the above rules, and the definitions of the circular functions, we have the following principles :

*The sine is positive in the first and second quadrants, and negative in the third and fourth.*

*The cosine is positive in the first and fourth quadrants, and negative in the second and third.*

*The versed-sine and the co-versed-sine are always positive.*

*The tangent and cotangent are positive in the first and third quadrants, and negative in the second and fourth.*

*The secant is positive in the first and fourth quadrants, and negative in the second and third.*

*The cosecant is positive in the first and second quadrants, and negative in the third and fourth.*

#### LIMITING VALUES OF THE CIRCULAR FUNCTIONS.

60. The limiting values of the circular functions are those values which they have at the beginning and end of the different quadrants. Their numerical values are discovered by following them as the arc increases from  $0^\circ$  around to  $360^\circ$ , and so on around through  $450^\circ$ ,  $540^\circ$ , &c. The signs of these values are determined by the principle, that *the sign of a varying magnitude up to the limit, is the sign at the limit*. For illustration, let us examine the limiting values of the sine and tangent.



If we suppose the arc to be 0, the sine will be 0; as the arc increases, the sine increases until the arc becomes equal to  $90^\circ$ , when the sine becomes equal to  $+1$ , which is its greatest possible value; as the arc increases from  $90^\circ$ , the sine goes on diminishing until the arc becomes equal to  $180^\circ$ , when the sine becomes equal to  $+0$ ; as the arc increases from  $180^\circ$ , the sine becomes negative, and goes on increasing numerically, but *decreasing algebraically*, until the arc becomes equal to  $270^\circ$ , when the sine becomes equal to  $-1$ , which is its least *algebraical* value; as the arc increases from  $270^\circ$ , the sine goes on decreasing numerically, but *increasing algebraically*, until the arc becomes  $360^\circ$ , when the sine becomes equal to  $-0$ . It is  $-0$ , for this value of the arc, in accordance with the principle of limits.

The tangent is 0 when the arc is 0, and increases till the arc becomes  $90^\circ$ , when the tangent is  $+\infty$ ; in passing through  $90^\circ$ , the tangent changes from  $+\infty$  to  $-\infty$ , and as the arc increases the tangent decreases, numerically, but increases algebraically, till the arc becomes equal to  $180^\circ$ , when the tangent becomes equal to  $-0$ ; from  $180^\circ$  to  $270^\circ$ , the tangent is again positive, and at  $270^\circ$  it becomes equal to  $+\infty$ ; from  $270^\circ$  to  $360^\circ$ , the tangent is again negative, and at  $360^\circ$  it becomes equal to  $-0$ .

If we still suppose the arc to increase after reaching  $360^\circ$ , the functions will again go through the same changes, that is, the functions of an arc are the same as the functions that are increased by  $360^\circ$ ,  $720^\circ$ , &c.

By discussing the limiting values of all the circular functions we are enabled to form the following table:

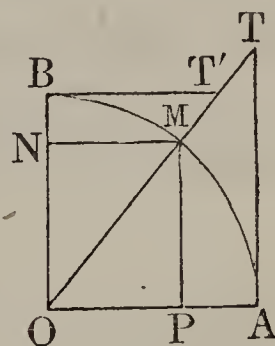
TABLE I.

Arc = 0.	Arc = 90°.	Arc = 180°.	Arc = 270°.	Arc = 360°.
sin = 0	sin = 1	sin = 0	sin = -1	sin = -0
cos = 1	cos = 0	cos = -1	cos = -0	cos = 1
v-sin = 0	v-sin = 1	v-sin = 2	v-sin = 1	v-sin = 0
co-v-sin = 1	co-v-sin = 0	co-v-sin = 1	co-v-sin = 2	c-v-sin = 1
tan = 0	tan = ∞	tan = -0	tan = ∞	tan = -0
cot = ∞	cot = 0	cot = -∞	cot = 0	cot = -∞
sec = 1	sec = ∞	sec = -1	sec = -∞	sec = 1
cosec = ∞	cosec = 1	cosec = ∞	cosec = -1	cosec = -∞

RELATIONS BETWEEN THE CIRCULAR FUNCTIONS OF ANY ARC.

61. Let  $AM$  represent any arc denoted by  $\alpha$ . Draw the lines as represented in the figure. Then we shall have, by definition

$$\begin{aligned}
 OM &= OA = 1; & PM &= ON = \sin \alpha; \\
 NM &= OP = \cos \alpha; & PA &= \text{ver-sin } \alpha; \\
 NB &= \text{co-ver-sin } \alpha; & AT &= \tan \alpha; \\
 BT' &= \cot \alpha; & OT &= \sec \alpha; \quad \text{and} \quad OT' = \text{cosec } \alpha.
 \end{aligned}$$



From the right-angled triangle  $OPM$ , we have,

$$\overline{PM}^2 + \overline{OP}^2 = \overline{OM}^2, \quad \text{or,} \quad \sin^2 \alpha + \cos^2 \alpha = 1. \quad (1.)$$

The symbols  $\sin^2 \alpha$ ,  $\cos^2 \alpha$ , &c., denote the square of the sine of  $\alpha$ , the square of the cosine of  $\alpha$ , &c.

From Formula (1) we have, by transposition,

$$\sin^2 \alpha = 1 - \cos^2 \alpha \quad (2); \quad \text{and} \quad \cos^2 \alpha = 1 - \sin^2 \alpha. \quad (3.)$$

We have, from the figure,

$$PA = OA - OP,$$

$$\text{or,} \quad \text{ver-sin } \alpha = 1 - \cos \alpha. \quad . \quad . \quad (4.)$$

$$\text{and,} \quad NB = OB - ON,$$

$$\text{or,} \quad \text{co-ver-sin } \alpha = 1 - \sin \alpha. \quad . \quad . \quad (5.)$$

From the similar triangles  $OAT'$  and  $OPM$ , we have,

$$OP : PM :: OA : AT, \quad \text{or,} \quad \cos \alpha : \sin \alpha :: 1 : \tan \alpha;$$

$$\text{whence,} \quad \tan \alpha = \frac{\sin \alpha}{\cos \alpha}. \quad . \quad . \quad . \quad . \quad . \quad (6.)$$

From the similar triangles  $ONM$  and  $OBT'$ , we have,

$$ON : NM :: OB : BT', \quad \text{or,} \quad \sin \alpha : \cos \alpha :: 1 : \cot \alpha;$$

$$\text{whence,} \quad \cot \alpha = \frac{\cos \alpha}{\sin \alpha}. \quad . \quad . \quad . \quad . \quad . \quad (7.)$$

Multiplying (6) and (7), member by member, we have,

$$\tan \alpha \cot \alpha = 1; \quad . \quad . \quad . \quad . \quad . \quad (8.)$$

whence, by division,

$$\tan \alpha = \frac{1}{\cot \alpha}; \quad . \quad (9.) \quad \text{and} \quad \cot \alpha = \frac{1}{\tan \alpha}. \quad . \quad (10.)$$

From the similar triangles  $OPM$  and  $OAT$ , we have,

$$OP : OM :: OA : OT, \quad \text{or,} \quad \cos \alpha : 1 :: 1 : \sec \alpha;$$

$$\text{whence,} \quad \sec \alpha = \frac{1}{\cos \alpha}. \quad . \quad . \quad . \quad . \quad . \quad (11.)$$



From the similar triangles  $ONM$  and  $OBT'$ , we have,

$$ON : OM :: OB : OT', \text{ or, } \sin a : 1 :: 1 : \text{co-sec } a;$$

whence, 
$$\text{co-sec } a = \frac{1}{\sin a} \cdot \cdot \cdot \cdot \cdot \cdot (12.)$$

From the right-angled triangle  $OAT$ , we have,

$$\overline{OT}^2 = \overline{OA}^2 + \overline{AT}^2; \text{ or, } \sec^2 a = 1 + \tan^2 a. \quad (13.)$$

From the right-angled triangle  $OBT'$ , we have,

$$\overline{OT'}^2 = \overline{OB}^2 + \overline{BT'}^2; \text{ or, } \text{co-sec}^2 a = 1 + \cot^2 a. \quad (14.)$$

It is to be observed that Formulas (5), (7), (12), and (14), may be deduced from Formulas (4), (6), (11), and (13), by substituting  $90^\circ - a$ , for  $a$ , and then making the proper reductions.

Collecting the preceding Formulas, we have the following table :

TABLE II.

(1.) $\sin^2 a + \cos^2 a = 1.$	(9.) $\tan a = \frac{1}{\cot a}.$
(2.) $\sin^2 a = 1 - \cos^2 a.$	(10.) $\cot a = \frac{1}{\tan a}.$
(3.) $\cos^2 a = 1 - \sin^2 a.$	(11.) $\sec a = \frac{1}{\cos a}.$
(4.) $\text{ver-sin } a = 1 - \cos a.$	(12.) $\text{cosec } a = \frac{1}{\sin a}.$
(5.) $\text{co-ver-sin } a = 1 - \sin a.$	(13.) $\sec^2 a = 1 + \tan^2 a.$
(6.) $\tan a = \frac{\sin a}{\cos a}.$	(14.) $\text{cosec}^2 a = 1 + \cot^2 a.$
(7.) $\cot a = \frac{\cos a}{\sin a}.$	
(8.) $\tan a \cot a = 1.$	

## FUNCTIONS OF NEGATIVE ARCS.

62. Let  $AM'''$ , estimated from  $A$  towards  $D$ , be numerically equal to  $AM$ ; then, if we denote the arc  $AM$  by  $a$ , the arc  $AM'''$  will be denoted by  $-a$  (Art. 48).

All the functions of  $AM'''$ , will be the same as those of  $ABM'''$ ; that is, the functions of  $-a$  are the same as the functions of  $360^\circ - a$ .

From an inspection of the figure, we shall discover the following relations, viz.:

$$\sin(-a) = -\sin a; \quad \cos(-a) = \cos a;$$

$$\tan(-a) = -\tan a; \quad \cot(-a) = -\cot a;$$

$$\sec(-a) = \sec a; \quad \operatorname{cosec}(-a) = -\operatorname{cosec} a.$$

## FUNCTIONS OF ARCS FORMED BY ADDING AN ARC TO, OR SUBTRACTING IT FROM ANY NUMBER OF QUADRANTS.

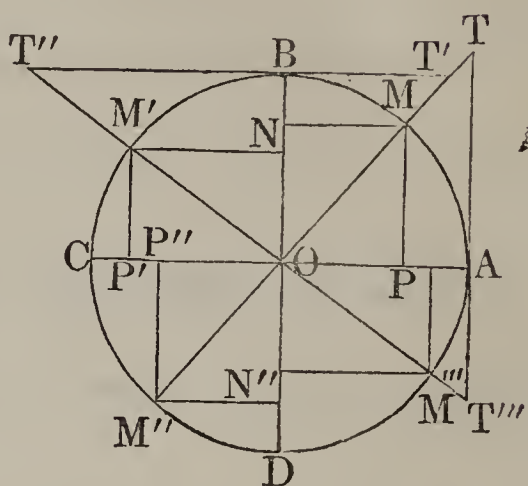
63. Let  $a$  denote any arc less than  $90^\circ$ . From what has preceded, we know that,

$$\sin(90^\circ - a) = \cos a; \quad \cos(90^\circ - a) = \sin a.$$

$$\tan(90^\circ - a) = \cot a; \quad \cot(90^\circ - a) = \tan a.$$

$$\sec(90^\circ - a) = \operatorname{cosec} a; \quad \operatorname{cosec}(90^\circ - a) = \sec a.$$

Now, suppose that  $BM' = a$ , then will  $AM' = 90^\circ + a$ . We see from the figure that,



$$NM' = \sin \alpha, \quad P'M' = \cos \alpha, \quad BT'' = \tan \alpha,$$

$$AT''' = \cot \alpha, \quad OT'' = \sec \alpha, \quad OT''' = \operatorname{cosec} \alpha,$$

without reference to their signs.

By a simple inspection of the figure, observing the rule for signs, we deduce the following relations :

$$\sin (90^\circ + \alpha) = \cos \alpha, \quad \cos (90^\circ + \alpha) = -\sin \alpha,$$

$$\tan (90^\circ + \alpha) = -\cot \alpha, \quad \cot (90^\circ + \alpha) = -\tan \alpha,$$

$$\sec (90^\circ + \alpha) = -\operatorname{cosec} \alpha, \quad \operatorname{cosec} (90^\circ + \alpha) = \sec \alpha.$$

Again, suppose

$$M'C = AM = \alpha; \quad \text{then will } \angle M' = 180^\circ - \alpha.$$

We see from the figure that,

$$P'M' = \sin \alpha, \quad OP' = \cos \alpha, \quad AT''' = \tan \alpha,$$

$$BT'' = \cot \alpha, \quad OT'' = \sec \alpha, \quad OT''' = \operatorname{cosec} \alpha,$$

without reference to their signs : hence, we have, as before, the following relations :

$$\sin (180^\circ - \alpha) = \sin \alpha, \quad \cos (180^\circ - \alpha) = -\cos \alpha,$$

$$\tan (180^\circ - \alpha) = -\tan \alpha, \quad \cot (180^\circ - \alpha) = -\cot \alpha,$$

$$\sec (180^\circ - \alpha) = -\sec \alpha, \quad \operatorname{cosec} (180^\circ - \alpha) = \operatorname{cosec} \alpha.$$

By a similar process, we may discuss the remaining arcs in question. Collecting the results, we have the following table :



TABLE III.

Arc = $90^\circ + a$ .		Arc = $270^\circ - a$ .	
sin =	cos $a$ ,	cos =	- sin $a$ ,
tan =	- cot $a$ ,	cot =	- tan $a$ ,
sec =	- cosec $a$ ,	cosec =	sec $a$ .
Arc = $180^\circ - a$ .		Arc = $270^\circ + a$ .	
sin =	sin $a$ ,	cos =	- cos $a$ ,
tan =	- tan $a$ ,	cot =	- cot $a$ ,
sec =	- sec $a$ ,	cosec =	cosec $a$ .
Arc = $180^\circ + a$ .		Arc = $360^\circ - a$ .	
sin =	- sin $a$ ,	cos =	- cos $a$ ,
tan =	tan $a$ ,	cot =	cot $a$ ,
sec =	- sec $a$ ,	cosec =	- cosec $a$ .

It will be observed that, when the arc is added to, or subtracted from, an *even* number of quadrants, the name of the function is the *same* in both columns; and when the arc is added to, or subtracted from, an *odd* number of quadrants, the names of the functions in the two columns are *contrary*: in all cases, the algebraic sign is determined by the rules already given (Art. 58).

By means of this table, we may find the functions of any arc in terms of the functions of an arc less than  $90^\circ$ . Thus,

$$\sin 115^\circ = \sin (90^\circ + 25^\circ) = \cos 25^\circ,$$

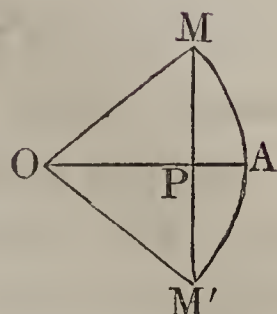
$$\sin 284^\circ = \sin (270^\circ + 14^\circ) = -\cos 14^\circ,$$

$$\sin 400^\circ = \sin (360^\circ + 40^\circ) = \sin 40^\circ,$$

$$\tan 210^\circ = \tan (180^\circ + 30^\circ) = \tan 30^\circ.$$

## PARTICULAR VALUES OF CERTAIN FUNCTIONS.

64. Let  $MAM'$  be any arc, denoted by  $2a$ ,  $M'M$  its chord, and  $OA$  a radius drawn perpendicular to  $M'M$ : then will  $PM = PM'$ , and  $AM = AM'$  (B. III., P. VI.). But  $PM$  is the sine of  $AM$ , or,  $PM = \sin a$ : hence,



$$\sin a = \frac{1}{2}M'M;$$

that is, *the sine of an arc is equal to one half the chord of twice the arc.*

Let  $M'AM = 60^\circ$ ; then will  $AM = 30^\circ$ , and  $M'M$  will equal the radius, or 1: hence, we have,

$$\sin 30^\circ = \frac{1}{2};$$

that is, *the sine of  $30^\circ$  is equal to half the radius.*

Also,

$$\cos 30^\circ = \sqrt{1 - \sin^2 30^\circ} = \frac{1}{2}\sqrt{3};$$

hence,

$$\tan 30^\circ = \frac{\sin 30^\circ}{\cos 30^\circ} = \sqrt{\frac{1}{3}}.$$

Again, let  $M'AM = 90^\circ$ : then will  $AM = 45^\circ$ , and  $M'M = \sqrt{2}$  (B. V., P. III.): hence, we have,

$$\sin 45^\circ = \frac{1}{2}\sqrt{2};$$

Also,

$$\cos 45^\circ = \sqrt{1 - \sin^2 45^\circ} = \frac{1}{2}\sqrt{2};$$

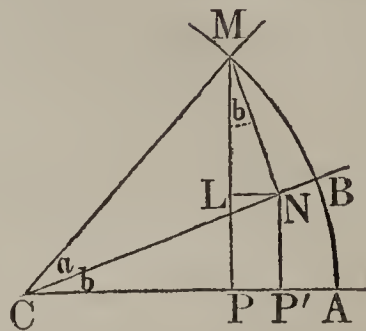
hence,

$$\tan 45^\circ = \frac{\sin 45^\circ}{\cos 45^\circ} = 1.$$

Many other numerical values might be deduced.

FORMULAS EXPRESSING RELATIONS BETWEEN THE CIRCULAR  
FUNCTIONS OF DIFFERENT ARCS.

65. Let  $AB$  and  $BM$  represent two arcs, having the common radius 1; denote the first by  $b$ , and the second by  $a$ . From  $M$  draw  $MP$ , perpendicular to  $CA$ , and  $MN$  perpendicular to  $CB$ ; from  $N$  draw  $NP'$  perpendicular to  $CA$ , and  $NL$  parallel to  $AC$ .



Then, by definition, we shall have,

$$PM = \sin(a + b), \quad NM = \sin a, \quad \text{and} \quad CN = \cos a.$$

From the figure, we have,

$$PM = PL + LM. \quad . \quad . \quad . \quad . \quad (1.)$$

From the right-angled triangle  $CP'N$  (Art. 37), we have,

$$P'N = CN \sin b;$$

or, since  $P'N = PL$ ,  $PL = \cos a \sin b$ .

Since the triangle  $MLN$  is similar to  $CP'N$ , the angle  $LMN$  is equal to the angle  $P'CN$ ; hence, from the right-angled triangle  $MLN$ , we have,

$$LM = MN \cos b = \sin a \cos b;$$

Substituting the values of  $PM$ ,  $PL$ , and  $LM$ , in Equation (1), we have,

$$\sin(a + b) = \sin a \cos b + \cos a \sin b; \quad . \quad (\Delta.)$$

that is, *the sine of the sum of two arcs, is equal to the sine of the first into the cosine of the second, plus the cosine of the first into the sine of the second.*



Since the above formula is true for any values of  $a$  and  $b$ , we may substitute  $-b$ , for  $b$ ; whence,

$$\sin (a - b) = \sin a \cos (-b) + \cos a \sin (-b);$$

but (Art. 62),

$$\cos (-b) = \cos b, \quad \text{and,} \quad \sin (-b) = -\sin b;$$

hence,

$$\sin (a - b) = \sin a \cos b - \cos a \sin b; \quad \cdot \quad (\text{B.})$$

that is, *the sine of the difference of two arcs, is equal to the sine of the first into the cosine of the second, minus the cosine of the first into the sine of the second.*

If, in Formula (B), we substitute  $(90^\circ - a)$ , for  $a$ , we have,

$$\sin (90^\circ - a - b) = \sin (90^\circ - a) \cos b - \cos (90^\circ - a) \sin b; \quad \cdot \quad (2.)$$

but (Art. 63),

$$\sin (90^\circ - a - b) = \sin [90^\circ - (a + b)] = \cos (a + b),$$

and,

$$\sin (90^\circ - a) = \cos a, \quad \cos (90^\circ - a) = \sin a;$$

hence, by substitution in Equation (2), we have,

$$\cos (a + b) = \cos a \cos b - \sin a \sin b; \quad \cdot \quad (\text{C.})$$

that is, *the cosine of the sum of two arcs, is equal to the rectangle of their cosines, minus the rectangle of their sines.*

If, in Formula (C), we substitute  $-b$ , for  $b$ , we find,

$$\cos (a - b) = \cos a \cos (-b) - \sin a \sin (-b),$$

or,

$$\cos (a - b) = \cos a \cos b + \sin a \sin b; \quad \cdot \quad (\text{D.})$$

that is, *the cosine of the difference of two arcs, is equal to the rectangle of their cosines, plus the rectangle of their sines.*

If we divide Formula (A) by Formula (C), member by member, we have,

$$\frac{\sin (a+b)}{\cos (a+b)} = \frac{\sin a \cos b + \cos a \sin b}{\cos a \cos b - \sin a \sin b}.$$

Dividing both terms of the second member by  $\cos a \cos b$ , recollecting that the sine divided by the cosine is equal to the tangent, we find,

$$\tan (a+b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}; \quad \cdot \cdot \cdot \cdot \quad (\text{E.})$$

that is, *the tangent of the sum of two arcs, is equal to the sum of their tangents, divided by 1 minus the rectangle of their tangents*

If, in Formula (E), we substitute  $-b$ , for  $b$ , recollecting that  $\tan (-b) = -\tan b$ , we have,

$$\tan (a-b) = \frac{\tan a - \tan b}{1 + \tan a \tan b}; \quad \cdot \cdot \cdot \cdot \quad (\text{F.})$$

that is, *the tangent of the difference of two arcs, is equal to the difference of their tangents, divided by 1 plus the rectangle of their tangents.*

In like manner, dividing Formula (C) by Formula (A), member by member, and reducing, we have,

$$\cot (a+b) = \frac{\cot a \cot b - 1}{\cot a + \cot b}; \quad \cdot \quad \cdot \quad \cdot \quad (\text{G.})$$

and thence, by the substitution of  $-b$ , for  $b$ ,

$$\cot (a - b) = \frac{\cot a \cot b + 1}{\cot b - \cot a} ; \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (\text{H.})$$

FUNCTIONS OF DOUBLE ARCS AND HALF ARCS.

66. If, in Formulas (A), (C), (E), and (G), we make  $a = b$ , we find,

$$\sin 2a = 2 \sin a \cos a ; \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (\text{A}').$$

$$\cos 2a = \cos^2 a - \sin^2 a ; \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (\text{C}').$$

$$\tan 2a = \frac{2 \tan a}{1 - \tan^2 a} ; \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (\text{E}').$$

$$\cot 2a = \frac{\cot^2 a - 1}{2 \cot a} \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (\text{G}').$$

Substituting in (C'), for  $\cos^2 a$ , its value,  $1 - \sin^2 a$ ; and afterwards for  $\sin^2 a$ , its value,  $1 - \cos^2 a$ , we have,

$$\cos 2a = 1 - 2 \sin^2 a,$$

$$\cos 2a = 2 \cos^2 a - 1 ;$$

whence, by solving these equations,

$$\sin a = \sqrt{\frac{1 - \cos 2a}{2}} ; \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (1.)$$

$$\cos a = \sqrt{\frac{1 + \cos 2a}{2}} \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (2.)$$

We also have, from the same equations,

$$1 - \cos 2a = 2 \sin^2 a ; \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (3.)$$

$$1 + \cos 2a = 2 \cos^2 a. \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (4.)$$



Dividing Equation (A'), first by Equation (4), and then by Equation (3), member by member, we have,

$$\frac{\sin 2a}{1 + \cos 2a} = \tan a; \quad \cdot \cdot \cdot \cdot \cdot \cdot \cdot (5.)$$

$$\frac{\sin 2a}{1 - \cos 2a} = \cot a. \quad \cdot \cdot \cdot \cdot \cdot \cdot \cdot (6.)$$

Substituting  $\frac{1}{2}a$ , for  $a$ , in Equations (1), (2), (5), and (6), we have,

$$\sin \frac{1}{2}a = \sqrt{\frac{1 - \cos a}{2}}; \quad \cdot \cdot \cdot \cdot (A'')$$

$$\cos \frac{1}{2}a = \sqrt{\frac{1 + \cos a}{2}}; \quad \cdot \cdot \cdot \cdot (B'')$$

$$\tan \frac{1}{2}a = \frac{\sin a}{1 + \cos a}; \quad \cdot \cdot \cdot \cdot (C'')$$

$$\cot \frac{1}{2}a = \frac{\sin a}{1 - \cos a} \cdot \cdot \cdot \cdot (D'')$$

Taking the reciprocals of both members of the last two formulas, we have also,

$$\cot \frac{1}{2}a = \frac{1 + \cos a}{\sin a}, \quad \text{and,} \quad \tan \frac{1}{2}a = \frac{1 - \cos a}{\sin a}.$$

#### ADDITIONAL FORMULAS.

67. If Formulas (A) and (B) be first added, member to member, and then subtracted, and the same operations be performed upon (C) and (D), we shall obtain,

$$\sin (a + b) + \sin (a - b) = 2 \sin a \cos b ;$$

$$\sin (a + b) - \sin (a - b) = 2 \cos a \sin b ;$$

$$\cos (a + b) + \cos (a - b) = 2 \cos a \cos b ;$$

$$\cos (a - b) - \cos (a + b) = 2 \sin a \sin b.$$

If in these we make,

$$a + b = p, \quad \text{and} \quad a - b = q,$$

whence,

$$a = \frac{1}{2} (p + q), \quad b = \frac{1}{2} (p - q) ;$$

and then substitute in the above formulas, we obtain,

$$\sin p + \sin q = 2 \sin \frac{1}{2} (p + q) \cos \frac{1}{2} (p - q) \quad \cdot \quad (\text{K.})$$

$$\sin p - \sin q = 2 \cos \frac{1}{2} (p + q) \sin \frac{1}{2} (p - q) \quad \cdot \quad (\text{L.})$$

$$\cos p + \cos q = 2 \cos \frac{1}{2} (p + q) \cos \frac{1}{2} (p - q) \quad \cdot \quad (\text{M.})$$

$$\cos q - \cos p = 2 \sin \frac{1}{2} (p + q) \sin \frac{1}{2} (p - q) \quad \cdot \quad (\text{N.})$$

From Formulas (L) and (K), by division, we obtain,

$$\frac{\sin p - \sin q}{\sin p + \sin q} = \frac{\cos \frac{1}{2} (p + q) \sin \frac{1}{2} (p - q)}{\sin \frac{1}{2} (p + q) \cos \frac{1}{2} (p - q)} = \frac{\tan \frac{1}{2} (p - q)}{\tan \frac{1}{2} (p + q)} \quad \cdot \quad (1.)$$

That is, *the sum of the sines of two arcs is to their difference, as the tangent of one half the sum of the arcs is to the tangent of one half their difference.*

Also, in like manner, we obtain,

$$\frac{\sin p + \sin q}{\cos p + \cos q} = \frac{\sin \frac{1}{2}(p+q) \cos \frac{1}{2}(p-q)}{\cos \frac{1}{2}(p+q) \cos \frac{1}{2}(p-q)} = \tan \frac{1}{2}(p+q) \quad (2.)$$

$$\frac{\sin p - \sin q}{\cos p + \cos q} = \frac{\sin \frac{1}{2}(p-q) \cos \frac{1}{2}(p+q)}{\cos \frac{1}{2}(p+q) \cos \frac{1}{2}(p-q)} = \tan \frac{1}{2}(p-q) \quad (3.)$$

$$\frac{\sin p + \sin q}{\sin (p+q)} = \frac{\sin \frac{1}{2}(p+q) \cos \frac{1}{2}(p-q)}{\sin \frac{1}{2}(p+q) \cos \frac{1}{2}(p+q)} = \frac{\cos \frac{1}{2}(p-q)}{\cos \frac{1}{2}(p+q)} \quad (4.)$$

$$\frac{\sin p - \sin q}{\sin (p+q)} = \frac{\sin \frac{1}{2}(p-q) \cos \frac{1}{2}(p+q)}{\sin \frac{1}{2}(p+q) \cos \frac{1}{2}(p+q)} = \frac{\sin \frac{1}{2}(p-q)}{\sin \frac{1}{2}(p+q)} \quad (5.)$$

$$\frac{\sin (p-q)}{\sin p - \sin q} = \frac{\sin \frac{1}{2}(p-q) \cos \frac{1}{2}(p-q)}{\sin \frac{1}{2}(p-q) \cos \frac{1}{2}(p+q)} = \frac{\cos \frac{1}{2}(p-q)}{\cos \frac{1}{2}(p+q)} \quad (6.)$$

all of which give proportions analogous to that deduced from Formula (1).

Since the second members of (6) and (4) are the same, we have,

$$\frac{\sin p - \sin q}{\sin (p-q)} = \frac{\sin (p+q)}{\sin p + \sin q} ; \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (7.)$$

That is, *the sine of the difference of two arcs is to the difference of the sines as the sum of the sines to the sine of the sum.*

All of the preceding formulas may be made homogeneous in terms of  $R$ ,  $R$  being any radius, as explained in Art. 30 ; or, we may simply introduce  $R$ , as a factor, into each term as many times as may be necessary to render all of its terms of the same degree.



## METHOD OF COMPUTING A TABLE OF NATURAL SINES.

68. Since the length of the semi-circumference of a circle whose radius is 1, is equal to the number 3.14159265 . . . , if we divide this number by 10800, the number of minutes in  $180^\circ$ , the quotient, .0002908882 . . . , will be the length of the arc of *one minute*; and since this arc is so small that it does not differ materially from its sine or tangent, this may be placed in the table as *the sine of one minute*.

Formula (3) of Table II., gives,

$$\cos 1' = \sqrt{1 - \sin^2 1'} = .9999999577 \quad . \quad . \quad (1.)$$

Having thus determined, to a near degree of approximation, the sine and cosine of one minute, we take the first formula of Art. 67, and put it under the form,

$$\sin (a + b) = 2 \sin a \cos b - \sin (a - b),$$

and make in this,  $b = 1'$ , and then in succession,

$$a = 1', \quad a = 2', \quad a = 3', \quad a = 4', \quad \&c.,$$

and obtain,

$$\sin 2' = 2 \sin 1' \cos 1' - \sin 0 = .0005817764 \dots$$

$$\sin 3' = 2 \sin 2' \cos 1' - \sin 1' = .0008726646 \dots$$

$$\sin 4' = 2 \sin 3' \cos 1' - \sin 2' = .0011635526 \dots$$

$$\sin 5' = \quad \quad \&c.,$$

thus obtaining the sine of every number of degrees and minutes from  $1'$  to  $45^\circ$ .

The cosines of the corresponding arcs may be computed by means of Equation (1).

Having found the sines and cosines of arcs less than  $45^\circ$ , those of the arcs between  $45^\circ$  and  $90^\circ$ , may be deduced, by considering that the sine of an arc is equal to the cosine of its complement, and the cosine equal to the sine of the complement. Thus,

$$\sin 50^\circ = \sin (90^\circ - 40^\circ) = \cos 40^\circ, \quad \cos 50^\circ = \sin 40^\circ,$$

in which the second members are known from the previous computations.

To find the tangent of any arc, divide its sine by its cosine. To find the cotangent, take the reciprocal of the corresponding tangent.

As the accuracy of the calculation of the sine of any arc, by the above method, depends upon the accuracy of each previous calculation, it would be well to verify the work, by calculating the sines of the degrees separately (after having found the sines of one and two degrees), by the last proportion of Art. 67. Thus,

$$\sin 1^\circ : \sin 2^\circ - \sin 1^\circ :: \sin 2^\circ + \sin 1^\circ : \sin 3^\circ ;$$

$$\sin 2^\circ : \sin 3^\circ - \sin 1^\circ :: \sin 3^\circ + \sin 1^\circ : \sin 4^\circ ; \&c.$$

# SPHERICAL TRIGONOMETRY.

---

69. SPHERICAL TRIGONOMETRY is that branch of Mathematics which treats of the solution of spherical triangles.

In every spherical triangle there are six parts: three sides and three angles. In general, any three of these parts being given, the remaining parts may be found.

## GENERAL PRINCIPLES.

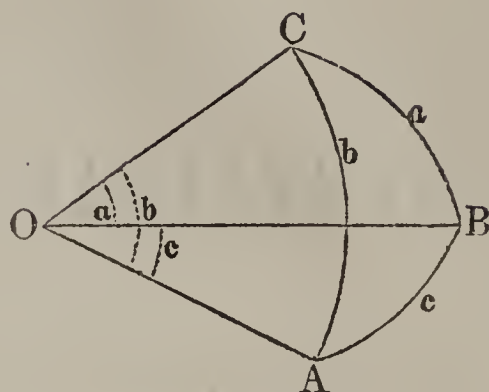
70. For the purpose of deducing the formulas required in the solution of spherical triangles, we shall suppose the triangles to be situated on spheres whose radii are equal to 1. The formulas thus deduced may be rendered applicable to triangles lying on any sphere, by making them homogeneous in terms of the radius of that sphere, as explained in Art. 30. The only cases considered will be those in which each of the sides and angles is less than  $180^\circ$ .

Any angle of a spherical triangle is the same as the dihedral angle included by the planes of its sides, and its measure is equal to that of the angle included between two right lines, one in each plane, and both perpendicular to their common intersection at the same point (B. VI., D. 4).

The radius of the sphere being equal to 1, each side of the triangle will measure the angle, at the centre, subtended by it. Thus, in the triangle  $ABC$ , the angle at  $A$  is



the same as that included between the planes  $AOB$  and  $BOC$ ; and the side  $a$  is the measure of the plane angle  $BOC$ ,  $O$  being the centre of the sphere, and  $OB$  the radius, equal to 1.

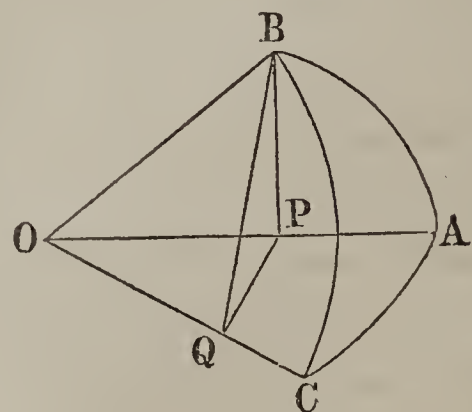


71. Spherical triangles, like plane triangles, are divided into two classes, *right-angled spherical triangles*, and *oblique-angled spherical triangles*. Each class will be considered in turn.

We shall, as before, denote the angles by the capital letters  $A$ ,  $B$ , and  $C$ , and the opposite sides by the small letters  $a$ ,  $b$ , and  $c$ .

#### FORMULAS USED IN SOLVING RIGHT-ANGLED SPHERICAL TRIANGLES.

72. Let  $CAB$  be a spherical triangle, right-angled at  $A$ , and let  $O$  be the centre of the sphere on which it is situated. Denote the angles of the triangle by the letters  $A$ ,  $B$ , and  $C$ , and the opposite sides by the letters  $a$ ,  $b$ , and  $c$ , recollecting that  $B$  and  $C$  may change places, provided that  $b$  and  $c$  change places at the same time.



Draw  $OA$ ,  $OB$ , and  $OC$ , each of which will be equal to 1. From  $B$ , draw  $BP$  perpendicular to  $OA$ , and from  $P$  draw  $PQ$  perpendicular to  $OC$ ; then join the points  $Q$  and  $B$ , by the line  $QB$ . The line  $QB$  will be perpendicular to  $OC$  (B. VI., P. VI.), and the angle  $PQB$

will be equal to the inclination of the planes  $OCB$  and  $OCA$ ; that is, it will be equal to the angle  $C$ .

We have, from the figure,

$$PB = \sin c, \quad OP = \cos c, \quad QB = \sin a, \quad OQ = \cos a.$$

From the right-angled triangles  $OQP$  and  $QPB$ , we have,

$$OQ = OP \cos AOC; \quad \text{or,} \quad \cos a = \cos c \cos b. \quad (1.)$$

$$PB = QB \sin PQB; \quad \text{or,} \quad \sin c = \sin a \sin C. \quad (2.)$$

If we multiply both terms of the fraction  $\frac{QP}{QB}$ , by  $OQ$ , we shall have,

$$\frac{QP}{QB} = \frac{OQ}{QB} \times \frac{QP}{OQ}; \quad \text{or,} \quad \cos C = \tan (90^\circ - a) \tan b. \quad (3.)$$

If we multiply both terms of the fraction  $\frac{QP}{OP}$ , by  $PB$ , we have,

$$\frac{QP}{OP} = \frac{PB}{OP} \times \frac{QP}{PB}; \quad \text{or,} \quad \sin b = \tan c \tan (90^\circ - C). \quad (4.)$$

If, in (2), we change  $c$  and  $C$ , into  $b$  and  $B$ , we have,

$$\sin b = \sin a \sin B. \quad (5.)$$

If, in (3), we change  $b$  and  $C$ , into  $c$  and  $B$ , we have,

$$\cos B = \tan (90^\circ - a) \tan c. \quad (6.)$$

If, in (4), we change  $b$ ,  $c$ , and  $C$ , into  $c$ ,  $b$ , and  $B$ , we have,

$$\sin c = \tan b \tan (90^\circ - B). \quad (7.)$$

Multiplying (4) by (7), member by member, we have,

$$\sin b \sin c = \tan b \tan c \tan (90^\circ - B) \tan (90^\circ - C).$$

Dividing both members by  $\tan b \tan c$ , we have,

$$\cos b \cos c = \tan (90^\circ - B) \tan (90^\circ - C);$$

and substituting for  $\cos b \cos c$ , its value,  $\cos a$ , taken from (1), we have,

$$\cos a = \tan (90^\circ - B) \tan (90^\circ - C) \quad \cdot \quad \cdot \quad (8.)$$

Formula (6) may be written under the form,

$$\cos B = \frac{\cos a \sin c}{\sin a \cos c}.$$

Substituting for  $\cos a$ , its value,  $\cos b \cos c$ , taken from (1), and reducing, we have,

$$\cos B = \frac{\cos b \sin c}{\sin a}.$$

Again, substituting for  $\sin c$ , its value,  $\sin a \sin C$ , taken from (2), and reducing, we have,

$$\cos B = \cos b \sin C \quad \cdot \quad \cdot \quad \cdot \quad (9.)$$

Changing  $B$ ,  $b$ , and  $C$ , in (9), into  $C$ ,  $c$ , and  $B$ , we have,

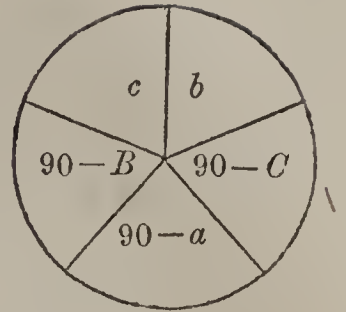
$$\cos C = \cos c \sin B \quad \cdot \quad \cdot \quad \cdot \quad (10.)$$

These ten formulas are sufficient for the solution of any right-angled spherical triangle whatever.



NAPIER'S CIRCUIAR PARTS.

73. *The two sides about the right angle, the complements of their opposite angles, and the complement of the hypotenuse, are called Napier's Circular Parts.*



If these parts be arranged in their order, as shown in the figure, we see that each part is adjacent to two of the others, and that it is separated from each of two remaining parts by an intervening part. If any part be taken as a *middle part*, those which are adjacent to it are called *adjacent parts*, and those which are separated from it, are called *opposite parts*. Thus,  $90^\circ - B$ , and  $90^\circ - C$ , are *adjacent parts* to  $90^\circ - a$ ; and  $c$  and  $b$  are *opposite parts*; and so on, for each of the other parts.

74. Formulas (1), (2), (5), (9), and (10), of Art. 72, may be written as follows:

$$\sin (90^\circ - a) = \cos b \cos c \cdot \cdot \cdot \cdot \cdot (1.)$$

$$\sin c = \cos (90^\circ - a) \cos (90^\circ - C) \cdot (2.)$$

$$\sin b = \cos (90^\circ - a) \cos (90^\circ - B) \cdot (3.)$$

$$\sin (90^\circ - B) = \cos b \cos (90^\circ - C) \cdot \cdot \cdot \cdot (4.)$$

$$\sin (90^\circ - C) = \cos c \cos (90^\circ - B) \cdot \cdot \cdot \cdot (5.)$$

Comparing these formulas with the figure, we see that,

*The sine of the middle part is equal to the rectangle of the cosines of the opposite parts.*

Formulas (8), (7), (4), (6), and (3), of Art. 72, may be written as follows:

$$\sin (90^\circ - a) = \tan (90^\circ - B) (\tan 90^\circ - C) \cdot \quad (6.)$$

$$\sin c = \tan b \tan (90^\circ - B) \cdot \cdot \cdot \cdot \quad (7.)$$

$$\sin b = \tan c \tan (90^\circ - C) \cdot \cdot \cdot \cdot \quad (8.)$$

$$\sin (90^\circ - B) = \tan (90^\circ - a) \tan c \cdot \cdot \cdot \cdot \quad (9.)$$

$$\sin (90^\circ - C) = \tan (90^\circ - a) \tan b \cdot \cdot \cdot \cdot \quad (10.)$$

Comparing these formulas with the figure, we see that,

*The sine of the middle part is equal to the rectangle of the tangents of the adjacent parts.*

These two rules are called Napier's rules for Circular Parts, and they are sufficient to solve any right-angled spherical triangle.

75. In applying Napier's rules for circular parts, the part sought will be determined by its sine. Now, the same sine corresponds to two different arcs, supplements of each other; it is, therefore, necessary to discover such relations between the given and required parts, as will serve to point out which of the two arcs is to be taken.

Two parts of a spherical triangle are said to be of *the same species*, when they are both less than  $90^\circ$ , or both greater than  $90^\circ$ ; and of *different species*, when one is less and the other greater than  $90^\circ$ .

From Formulas (9) and (10), Art. 72, we have,

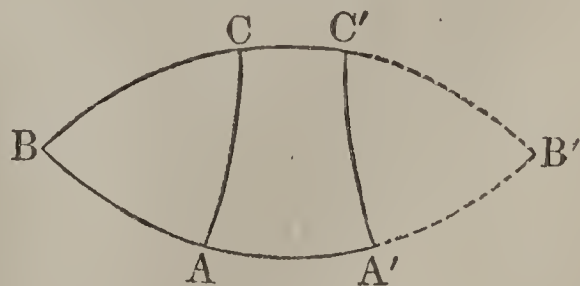
$$\sin C = \frac{\cos B}{\cos b}, \quad \text{and} \quad \sin B = \frac{\cos C}{\cos c};$$

since the angles  $B$  and  $C$  are both less than  $180^\circ$ , their sines must always be positive : hence,  $\cos B$  must have the same sign as  $\cos b$ , and the  $\cos C$  must have the same sign as  $\cos c$ . This can only be the case when  $B$  is of the same species as  $b$ , and  $C$  of the same species as  $c$  ; that is, *the sides about the right angle are always of the same species as their opposite angles.*

From Formula (1), we see that when  $a$  is less than  $90^\circ$ , or when  $\cos a$  is positive, the cosines of  $b$  and  $c$  will have the same sign ; that is,  $b$  and  $c$  will be of the same species. When  $a$  is greater than  $90^\circ$ , or when  $\cos a$  is negative, the cosines of  $b$  and  $c$  will be contrary ; that is,  $b$  and  $c$  will be of different species : hence, *when the hypotenuse is less than  $90^\circ$ , the two sides about the right angle, and consequently the two oblique angles, will be of the same species ; when the hypotenuse is greater than  $90^\circ$ , the two sides about the right angle, and consequently the two oblique angles, will be of different species.*

These two principles enable us to determine the nature of the part sought, in every case, except when an oblique angle and the opposite side are given, to find the remaining parts. In this case, there may be *two solutions, one solution, or no solution at all.*

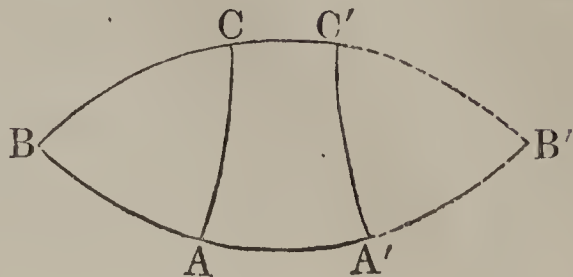
Let  $BAC$  be a right-angled triangle, in which  $B$  and  $b$  are given. Prolong the sides  $BA$  and  $BC$  till they meet in  $B'$ . Take



$B'A' = BA$ ,  $B'C' = BC$ , and join  $A'$  and  $C'$  by the arc of a great circle : then, because the triangles  $BAC$  and  $B'A'C'$  have two sides and the included angle of the one, equal to two sides and the included angle of the other, each to each, the remaining parts will be equal, each to each ;



that is,  $A'C' = AC$ , and the angle  $A'$  equal to the angle  $A$  : hence, the two triangles  $BAC$ ,  $B'A'C'$ , are right-angled ; they have also one oblique angle and the opposite side, in each, equal.



Now, if  $b$  differs *more* from  $90^\circ$  than  $B$ , there will evidently be *two solutions*, the sides

including the given angle, in the one case, being supplements of those which include the given angle, in the other case.

If  $b = B$ , the triangle will be bi-rectangular, and there will be but a *single solution*.

If  $b$  differs less from  $90^\circ$  than  $B$ , the triangle cannot be constructed, that is, there will be *no solution*.

#### SOLUTION OF RIGHT-ANGLED SPHERICAL TRIANGLES.

76. In a right-angled spherical triangle, the right angle is always known. If any two of the other parts are given, the remaining parts may be found by Napier's rules for circular parts. Six cases may arise. There may be given,

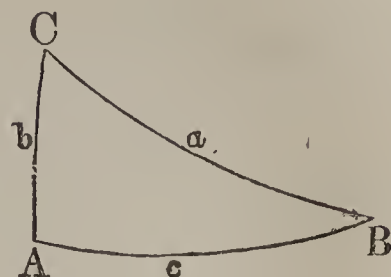
- I. The hypotenuse and one side.
- II. The hypotenuse and one oblique angle.
- III. The two sides about the right angle.
- IV. One side and its adjacent angle.
- V. One side and its opposite angle.
- VI. The two oblique angles.

In any one of these cases, we select that part which is either adjacent to, or separated from, each of the other given parts, and calling it the middle part, we employ that one of Napier's rules which is applicable. Having determined a third part, the other two may then be found in a similar manner.

It is to be observed, that the formulas employed are to be rendered homogeneous, in terms of  $R$ , as explained in Art. 30. The method of proceeding will be readily understood from a few examples.

## EXAMPLES.

1. Given  $a = 105^\circ 17' 29''$ , and  $b = 38^\circ 47' 11''$ , to find  $c$ ,  $B$ , and  $C$ .



Since  $a > 90^\circ$ ,  $b$  and  $c$  must be of different species, that is,  $c > 90^\circ$ ; for the same reason,  $C > 90^\circ$ .

## OPERATION.

From Formula (10), Art. 74, we have,

$$\begin{aligned} \log \cos C &= \log \cot a + \log \tan b - 10; \\ \log \cot a \quad (105^\circ 17' 29'') & 9.436811 \\ \log \tan b \quad (38^\circ 47' 11'') & 9.905055 \\ \log \cos C \quad . \quad . \quad . & \underline{9.341866} \quad \therefore C = 102^\circ 41' 33''. \end{aligned}$$

From Formula (2), Art. 74, we have,

$$\begin{aligned} \log \sin c &= \log \sin a + \log \sin C - 10; \\ \log \sin a \quad (105^\circ 17' 29'') & 9.984346 \\ \log \sin C \quad (102^\circ 41' 33'') & 9.989256 \\ \log \sin c \quad . \quad . \quad . & \underline{9.973602} \quad \therefore c = 109^\circ 46' 32''. \end{aligned}$$

From Formula (4), we have,

$$\begin{aligned} \log \cos B &= \log \sin C + \log \cos b - 10; \\ \log \sin C \quad (102^\circ 41' 33'') & 9.989256 \\ \log \cos b \quad (38^\circ 47' 11'') & 9.891808 \\ \log \cos B \quad . \quad . \quad . & \underline{9.881064} \quad \therefore B = 40^\circ 29' 50''. \end{aligned}$$

*Ans.*  $c = 109^\circ 46' 32''$ ,  $B = 40^\circ 29' 50''$ ,  $C = 102^\circ 41' 33''$ .

2. Given  $b = 51^\circ 30'$ , and  $B = 58^\circ 35'$ , to find  $a$ ,  $c$ , and  $C$ .

Because  $b < B$ , there are two solutions.

#### OPERATION.

From Formula (7), we have,

$$\log \sin c = \log \tan b + \log \cot B - 10;$$

log tan $b$	( $51^\circ 30'$ )	.	10.099395	
log cot $B$	( $58^\circ 35'$ )	.	<u>9.785900</u>	
log sin $c$	.	.	<u>9.885295</u>	$\therefore c = 50^\circ 09' 51''$ ,
				and $c = 129^\circ 50' 09''$ .

From Formula (1), we have,

$$\log \cos a = \log \cos b + \log \cos c - 10;$$

log cos $b$	( $51^\circ 30'$ )	.	9.794150	
log cos $c$	( $50^\circ 09' 51''$ )	.	<u>9.806580</u>	
log cos $a$	.	.	<u>9.600730</u>	$\therefore a = 66^\circ 29' 54''$ ,
				and $a = 113^\circ 30' 06''$ .

From Formula (10), we have,

$$\log \cos C = \log \tan b + \log \cot a - 10;$$

log tan $b$	( $51^\circ 30'$ )	.	10.099395	
log cot $a$	( $66^\circ 29' 54''$ )	.	<u>9.638336</u>	
log cos $C$	.	.	<u>9.737731</u>	$\therefore C = 56^\circ 51' 38''$ ,
				and $C = 123^\circ 08' 22''$ .

In a similar manner, all other cases may be solved.

3. Given  $a = 86^\circ 51'$ , and  $B = 18^\circ 03' 32''$ , to find  $b$ ,  $c$ , and  $C$ .

*Ans.*  $b = 18^\circ 01' 50''$ ,  $c = 86^\circ 41' 14''$ ,  $C = 88^\circ 58' 25''$ .



4. Given  $b = 155^\circ 27' 54''$ , and  $c = 29^\circ 46' 08''$ , to find  $a$ ,  $B$ , and  $C$ .

*Ans.*  $a = 142^\circ 09' 13''$ ,  $B = 137^\circ 24' 21''$ ,  $C = 54^\circ 01' 16''$ .

5. Given  $c = 73^\circ 41' 35''$ , and  $B = 99^\circ 17' 33''$ , to find  $a$ ,  $b$ , and  $C$ .

*Ans.*  $a = 92^\circ 42' 17''$ ,  $b = 99^\circ 40' 30''$ ,  $C = 73^\circ 54' 47''$ .

6. Given  $b = 115^\circ 20'$ , and  $B = 91^\circ 01' 47''$ , to find  $a$ ,  $c$ , and  $C$ .

$$a = \begin{cases} 64^\circ 41' 11'', \\ 115^\circ 18' 49'', \end{cases} \quad c = \begin{cases} 177^\circ 49' 27'', \\ 2^\circ 10' 33'', \end{cases} \quad C = \begin{cases} 177^\circ 35' 36'', \\ 2^\circ 24' 24''. \end{cases}$$

7. Given  $B = 47^\circ 13' 48''$ , and  $C = 126^\circ 40' 24''$ , to find  $a$ ,  $b$ , and  $c$ .

*Ans.*  $a = 133^\circ 32' 26'$ ,  $b = 32^\circ 08' 56''$ ,  $c = 144^\circ 27' 03''$ .

In certain cases, it may be necessary to find but a single part. This may be effected, either by one of the formulas given in Art. 74, or by a slight transformation of one of them.

Thus, let  $a$  and  $B$  be given, to find  $C$ . Regarding  $90^\circ - a$ , as a middle part, we have,

$$\cos a = \cot B \cot C;$$

whence,

$$\cot C = \frac{\cos a}{\cot B};$$

and, by the application of logarithms,

$$\log \cot C = \log \cos a + (\text{a. c.}) \log \cot B;$$

from which  $C$  may be found. In like manner, other cases may be treated.

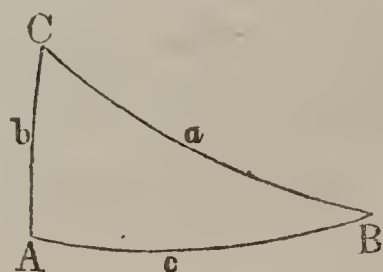
## QUADRANTAL SPHERICAL TRIANGLES.

77. A QUADRANTAL SPHERICAL TRIANGLE is one in which one side is equal to  $90^\circ$ . To solve such a triangle, we pass to its polar triangle, by subtracting each side and each angle from  $180^\circ$  (B. IX., P. VI.). The resulting polar triangle will be right-angled, and may be solved by the rules already given. The polar triangle of any quadrantal triangle being solved, the parts of the given triangle may be found by subtracting each part of the polar triangle from  $180^\circ$ .

## EXAMPLE.

Let  $A'B'C'$  be a quadrantal triangle, in which  $B'C' = 90^\circ$ ,  $B' = 75^\circ 42'$ , and  $c' = 18^\circ 37'$ .

Passing to the polar triangle, we have,



$$A = 90^\circ, \quad b = 104^\circ 18', \quad \text{and} \quad C = 161^\circ 23'.$$

Solving this triangle by previous rules, we find,

$$a = 76^\circ 25' 11'', \quad c = 161^\circ 55' 20'', \quad B = 94^\circ 31' 21'';$$

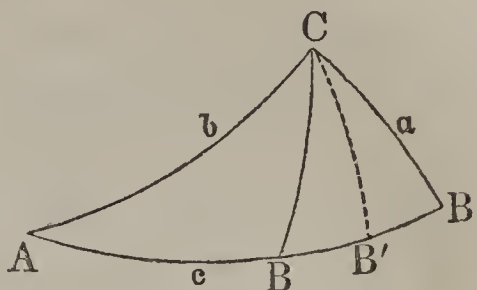
hence, the required parts of the given quadrantal triangle are,

$$A' = 103^\circ 34' 49'', \quad C' = 18^\circ 04' 40'', \quad b' = 85^\circ 28' 39''.$$

In a similar manner, other quadrantal triangles may be solved.

## FORMULAS USED IN SOLVING OBLIQUE-ANGLED SPHERICAL TRIANGLES.

78. Let  $ABC$  represent an oblique-angled spherical triangle. From either vertex,  $C$ , draw the arc of a great circle  $CB'$ , perpendicular to the opposite side. The two triangles  $ACB'$  and  $BCB'$  will be right-angled at  $B'$ .



From the triangle  $ACB'$ , we have Formula (2), Art. 74,

$$\sin CB' = \sin A \sin b.$$

From the triangle  $BCB'$ , we have,

$$\sin CB' = \sin B \sin a.$$

Equating these values of  $\sin CB'$ , we have,

$$\sin A \sin b = \sin B \sin a;$$

from which results the proportion,

$$\sin a : \sin b :: \sin A : \sin B \quad . \quad . \quad . \quad (1.)$$

In like manner, we may deduce,

$$\sin a : \sin c :: \sin A : \sin C \quad . \quad . \quad . \quad (2.)$$

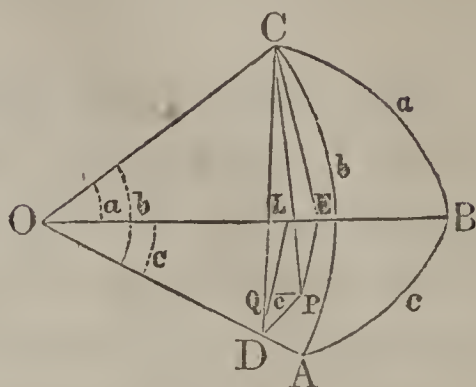
$$\sin b : \sin c :: \sin B : \sin C \quad . \quad . \quad . \quad (3.)$$

That is, in any spherical triangle, *the sines of the sides are proportional to the sines of their opposite angles.*

Had the perpendicular fallen on the prolongation of  $AB$ , the same relation would have been found.



79. Let  $ABC$  represent any spherical triangle, and  $O$  the centre of the sphere on which it is situated. Draw the radii  $OA$ ,  $OB$ , and  $OC$ ; from  $C$  draw  $CP$  perpendicular to the plane  $AOB$ ; from  $P$ , the foot of this perpendicular, draw  $PD$  and  $PE$  respectively perpendicular to  $OA$  and  $OB$ ; join  $CD$  and  $CE$ , these lines will be respectively perpendicular to  $OA$  and  $OB$  (B. VI., P. VI.), and the angles  $CDP$  and  $CEP$  will be equal to the angles  $A$  and  $B$  respectively. Draw  $DL$  and  $PQ$ , the one perpendicular, and the other parallel to  $OB$ . We then have,



$$OE = \cos a, \quad DC = \sin b, \quad OD = \cos b.$$

We have from the figure,

$$OE = OL + QP \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (1.)$$

In the right-angled triangle  $OLD$ ,

$$OL = OD \cos DOL = \cos b \cos c.$$

The right-angled triangle  $PQD$  has its sides respectively perpendicular to those of  $OLD$ ; it is, therefore, similar to it, and the angle  $QDP$  is equal to  $c$ , and we have,

$$QP = PD \sin QDP = PD \sin c \quad \cdot \quad \cdot \quad \cdot \quad (2.)$$

The right-angled triangle  $CPD$  gives,

$$PD = CD \cos CDP = \sin b \cos A;$$

substituting this value in (2), we have,

$$QP = \sin b \sin c \cos A;$$

and now substituting these values of  $OE$ ,  $OL$ , and  $QP$ , in (1), we have,

$$\cos a = \cos b \cos c + \sin b \sin c \cos A \quad \cdot \cdot \quad (3.)$$

In the same way, we may deduce,

$$\cos b = \cos a \cos c + \sin a \sin c \cos B \quad \cdot \cdot \quad (4.)$$

$$\cos c = \cos a \cos b + \sin a \sin b \cos C \quad \cdot \cdot \quad (5.)$$

That is, *the cosine of either side of a spherical triangle is equal to the rectangle of the cosines of the other two sides plus the rectangle of the sines of these sides into the cosine of their included angle.*

80. If we represent the angles of the polar triangle of  $ABC$ , by  $A'$ ,  $B'$ , and  $C'$ , and the sides by  $a'$ ,  $b'$ , and  $c'$ , we have (B. IX., P. VI.),

$$a = 180^\circ - A', \quad b = 180^\circ - B', \quad c = 180^\circ - C',$$

$$A = 180^\circ - a', \quad B = 180^\circ - b', \quad C = 180^\circ - c'.$$

Substituting these values in Equation (3), of the preceding article, and recollecting that,

$$\cos (180^\circ - A') = -\cos A', \quad \sin (180^\circ - B') = \sin B', \quad \&c.,$$

we have,

$$-\cos A' = \cos B' \cos C' - \sin B' \sin C' \cos a';$$

or, changing the signs and omitting the primes (since the preceding result is true for any triangle),

$$\cos A = \sin B \sin C \cos a - \cos B \cos C \quad \cdot \quad (1.)$$

In the same way, we may deduce,

$$\cos B = \sin A \sin C \cos b - \cos A \cos C . \quad (2.)$$

$$\cos C = \sin A \sin B \cos c - \cos A \cos B . \quad (3.)$$

That is, *the cosine of either angle of a spherical triangle is equal to the rectangle of the sines of the other two angles into the cosine of their included side, minus the rectangle of the cosines of these angles.*

81. From Equation (3), Art. 79, we deduce,

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c} . . . . . (1.)$$

If we add this equation, member by member, to the number 1, and recollect that  $1 + \cos A$ , in the first member, is equal to  $2 \cos^2 \frac{1}{2}A$  (Art. 66), and reduce, we have,

$$2 \cos^2 \frac{1}{2}A = \frac{\sin b \sin c + \cos a - \cos b \cos c}{\sin b \sin c} ;$$

or, Formula (©), Art. 66,

$$2 \cos^2 \frac{1}{2}A = \frac{\cos a - \cos (b + c)}{\sin b \sin c} . . . . . (2.)$$

And since, Formula (N), Art. 67,

$$\cos a - \cos (b + c) = 2 \sin \frac{1}{2}(a + b + c) \sin \frac{1}{2}(b + c - a),$$

Equation (2) becomes, after dividing both members by 2,

$$\cos^2 \frac{1}{2}A = \frac{\sin \frac{1}{2}(a + b + c) \sin \frac{1}{2}(b + c - a)}{\sin b \sin c} .$$



If, in this we make,

$$\frac{1}{2}(a + b + c) = \frac{1}{2}s ; \quad \text{whence,} \quad \frac{1}{2}(b + c - a) = \frac{1}{2}s - a,$$

and extract the square root of both members, we have,

$$\cos \frac{1}{2}A = \sqrt{\frac{\sin \frac{1}{2}s \sin (\frac{1}{2}s - a)}{\sin b \sin c}} . . . . (3.)$$

That is, *the cosine of one-half of either angle of a spherical triangle, is equal to the square root of the sine of one-half of the sum of the three sides, into the sine of one-half this sum minus the side opposite the angle, divided by the rectangle of the sines of the adjacent sides.*

If we subtract Equation (1), of the preceding article, member by member, from the number 1, and recollect that,

$$1 - \cos A = 2 \sin^2 \frac{1}{2}A,$$

we find, after reduction,

$$\sin \frac{1}{2}A = \sqrt{\frac{\sin (\frac{1}{2}s - b) \sin (\frac{1}{2}s - c)}{\sin b \sin c}} . . . (4.)$$

Dividing the preceding value of  $\sin \frac{1}{2}A$ , by  $\cos \frac{1}{2}A$ , we obtain,

$$\tan \frac{1}{2}A = \sqrt{\frac{\sin (\frac{1}{2}s - b) \sin (\frac{1}{2}s - c)}{\sin \frac{1}{2}s \sin (\frac{1}{2}s - a)}} . . . (5.)$$

82. If the angles and sides of the polar triangle of  $ABC$  be represented as in Art. 80, we have;

$$A = 180^\circ - a', \quad b = 180^\circ - B', \quad c = 180^\circ - C',$$

$$\frac{1}{2}s = 270^\circ - \frac{1}{2}(A' + B' + C'), \quad \frac{1}{2}s - a = 90^\circ - \frac{1}{2}(B' + C' - A').$$

Substituting these values in (3), Art. 81, and reducing by the aid of the formulas in Table III., Art. 63, we find,

$$\sin \frac{1}{2}a' = \sqrt{\frac{-\cos \frac{1}{2}(A'+B'+C') \cos \frac{1}{2}(B'+C'-A')}{\sin B' \sin C'}}$$

Placing

$$\frac{1}{2}(A'+B'+C') = \frac{1}{2}S; \quad \text{whence,} \quad \frac{1}{2}(B'+C'-A') = \frac{1}{2}S - A'.$$

Substituting and omitting the primes, we have,

$$\sin \frac{1}{2}a = \sqrt{\frac{-\cos \frac{1}{2}S \cos (\frac{1}{2}S - A)}{\sin B \sin C}} \quad \cdot \quad \cdot \quad \cdot \quad (1.)$$

In a similar way, we may deduce from (4), Art. 81.

$$\cos \frac{1}{2}a = \sqrt{\frac{\cos (\frac{1}{2}S - B) \cos (\frac{1}{2}S - C)}{\sin B \sin C}} \quad \cdot \quad \cdot \quad (2.)$$

and thence,

$$\tan \frac{1}{2}a = \sqrt{\frac{-\cos \frac{1}{2}S \cos (\frac{1}{2}S - A)}{\cos (\frac{1}{2}S - B) \cos (\frac{1}{2}S - C)}} \quad \cdot \quad \cdot \quad \cdot \quad (3.)$$

83. From Equation (1), Art. 80, we have,

$$\cos A + \cos B \cos C = \sin B \sin C \cos a = \sin C \frac{\sin A}{\sin a} \sin b \cos a; \quad (1.)$$

since, from Proportion (1), Art. 78, we have,

$$\sin B = \frac{\sin A}{\sin a} \sin b.$$

Also, from Equation (2), Art. 80, we have,

$$\cos B + \cos A \cos C = \sin A \sin C \cos b = \sin C \frac{\sin A}{\sin a} \sin a \cos b. \quad (2.)$$

Adding (1) and (2), and dividing by  $\sin C$ , we obtain,

$$(\cos A + \cos B) \frac{1 + \cos C}{\sin C} = \frac{\sin A}{\sin a} \sin(a + b). \quad (3.)$$

The proportion,  $\sin A : \sin B :: \sin a : \sin b$ ,

taken first by composition, and then by division, gives,

$$\sin A + \sin B = \frac{\sin A}{\sin a} (\sin a + \sin b) \cdot \cdot \cdot \quad (4.)$$

$$\sin A - \sin B = \frac{\sin A}{\sin a} (\sin a - \sin b) \cdot \cdot \cdot \quad (5.)$$

Dividing (4) and (5), in succession, by (3), we obtain,

$$\frac{\sin A + \sin B}{\cos A + \cos B} \times \frac{\sin C}{1 + \cos C} = \frac{\sin a + \sin b}{\sin(a + b)} \cdot \cdot \cdot \quad (6.)$$

$$\frac{\sin A - \sin B}{\cos A + \cos B} \times \frac{\sin C}{1 + \cos C} = \frac{\sin a - \sin b}{\sin(a + b)} \cdot \cdot \cdot \quad (7.)$$

But, by Formulas (2) and (4), Art. 67, and Formula (E''), Art. 66, Equation (6) becomes,

$$\tan \frac{1}{2}(A + B) = \cot \frac{1}{2}C \frac{\cos \frac{1}{2}(a - b)}{\cos \frac{1}{2}(a + b)}; \cdot \cdot \cdot \quad (8.)$$

and, by the similar Formulas (3) and (5), of Art. 67, Equation (7) becomes,

$$\tan \frac{1}{2}(A - B) = \cot \frac{1}{2}C \frac{\sin \frac{1}{2}(a - b)}{\sin \frac{1}{2}(a + b)} \cdot \cdot \cdot \quad (9.)$$

These last two formulas give the proportions known as *the first set of Napier's Analogies*.

$$\cos \frac{1}{2}(a + b) : \cos \frac{1}{2}(a - b) :: \cot \frac{1}{2}C : \tan \frac{1}{2}(A + B). \quad (10.)$$

$$\sin \frac{1}{2}(a + b) : \sin \frac{1}{2}(a - b) :: \cot \frac{1}{2}C : \tan \frac{1}{2}(A - B). \quad (11.)$$



If in these we substitute the values of  $a$ ,  $b$ ,  $C$ ,  $A$ , and  $B$ , in terms of the corresponding parts of the polar triangle, as expressed in Art. 80, we obtain,

$$\cos \frac{1}{2}(A+B) : \cos \frac{1}{2}(A-B) :: \tan \frac{1}{2}c : \tan \frac{1}{2}(a+b). \quad (12.)$$

$$\sin \frac{1}{2}(A+B) : \sin \frac{1}{2}(A-B) :: \tan \frac{1}{2}c : \tan \frac{1}{2}(a-b). \quad (13.)$$

*the second set of Napier's Analogies.*

In applying logarithms to any of the preceding formulas, they must be made homogeneous, in terms of  $R$ , as explained in Art. 30.

#### SOLUTION OF OBLIQUE-ANGLED SPHERICAL TRIANGLES.

84. In the solution of oblique-angled triangles six different cases may arise: viz., there may be given,

- I. Two sides and an angle opposite one of them.
- II. Two angles and a side opposite one of them.
- III. Two sides and their included angle.
- IV. Two angles and their included side.
- V. The three sides.
- VI. The three angles.

#### CASE I.

*Given two sides and an angle opposite one of them.*

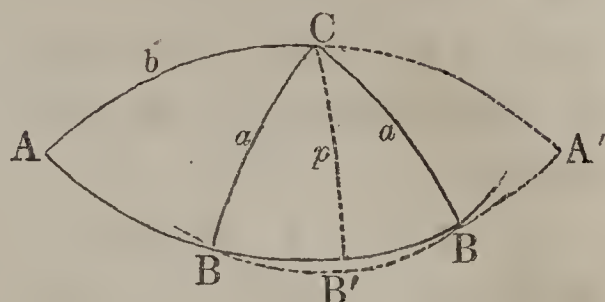
85. The solution, in this case, is commenced by finding the angle opposite the second given side, for which purpose Formula (1), Art. 78, is employed.

As this angle is found by means of its sine, and because the same sine corresponds to two different arcs, there would seem to be two different solutions. To ascertain when there are *two solutions*, when *one solution*, and when *no solution* at all, it becomes necessary to examine the relations which

may exist between the given parts. Two cases may arise, viz., the given angle may be *acute*, or it may be *obtuse*.

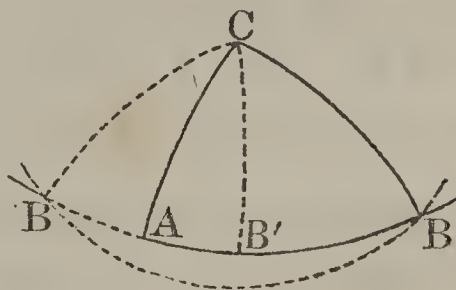
We shall consider each case separately (B. IX., P. XIX., Gen. Scholium).

*First Case.* Let  $A$  be the given angle, and let  $a$  and  $b$  be the given sides. Prolong the arcs  $AC$  and  $AB$  till they meet at  $A'$ , forming the lune  $AA'$ ; and



from  $C$ , draw the arc  $CB'$  perpendicular to  $ABA'$ . From  $C$ , as a pole, and with the arc  $a$ , describe the arc of a small circle  $BB$ . If this circle cuts  $ABA'$ , in two points between  $A$  and  $A'$ , there will be *two solutions*; for if  $C$  be joined with each point of intersection by the arc of a great circle, we shall have two triangles  $ABC$ , both of which will conform to the conditions of the problem.

If only one point of intersection lies between  $A$  and  $A'$ , or if the small circle is tangent to  $ABA'$ , there will be but *one solution*.



If there is no point of intersection, or if there are points of intersection which do not lie between  $A$  and  $A'$ , there will be *no solution*.

From Formula (2), Art. 72, we have,

$$\sin CB' = \sin b \sin A,$$

from which the perpendicular, which will be less than  $90^\circ$ , will be found. Denote its value by  $p$ . By inspection of the figure, we find the following relations:

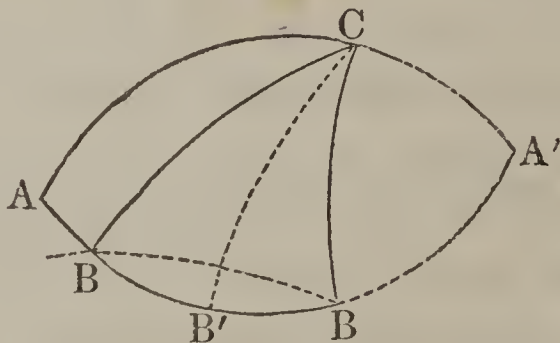
1. When  $a$  is greater than  $p$ , and at the same time less than both  $b$  and  $180^\circ - b$ , there will be two solutions.

2. When  $a$  is greater than  $p$ , and intermediate in value between  $b$  and  $180^\circ - b$ ; or, when  $a$  is equal to  $p$ , there will be but one solution.

If  $a = b$ , and is also less than  $180^\circ - b$ , one of the points of intersection will be at  $A$ , and there will be but one solution.

3. When  $a$  is greater than  $p$ , and at the same time greater than both  $b$  and  $180^\circ - b$ ; or, when  $a$  is less than  $p$ , there will be no solution.

*Second Case.* Adopt the same construction as before. In this case, the perpendicular will be greater than  $90^\circ$ , and greater also than any other arc  $CA$ ,  $CB$ ,  $CA'$ , that can be drawn from  $C$



to  $ABA'$ . By a course of reasoning entirely analogous to that in the preceding case, we have the following principles:

4. When  $a$  is less than  $p$ , and at the same time greater than both  $b$  and  $180^\circ - b$ , there will be two solutions.

5. When  $a$  is less than  $p$ , and intermediate in value between  $b$  and  $180^\circ - b$ ; or, when  $a$  is equal to  $p$ , there will be but one solution.

6. When  $a$  is less than  $p$ , and at the same time less than both  $b$  and  $180^\circ - b$ ; or, when  $a$  is greater than  $p$ , there will be no solution.

Having found the angle or angles opposite the second side, the solution may be completed by means of Napier's Analogies.



## EXAMPLES.

1. Given  $a = 43^\circ 27' 36''$ ,  $b = 82^\circ 58' 17''$ , and  $A = 29^\circ 32' 29''$ , to find  $B$ ,  $C$ , and  $c$ .

We see at a glance, that  $a > p$ , since  $p$  cannot exceed  $A$ ; we see further, that  $a$  is less than both  $b$  and  $180^\circ - b$ ; hence, from the first condition there will be two solutions.

Applying logarithms to Formula (1), Art. 78, we have,

$$\log \sin B = \log \sin b + \log \sin A + (\text{a. c.}) \log \sin a - 10;$$

$\log \sin b$	$\cdot \cdot (82^\circ 58' 17'')$	$\cdot \cdot \cdot$	9.996724
$\log \sin A$	$\cdot \cdot (29^\circ 32' 29'')$	$\cdot \cdot \cdot$	9.692893
(a. c.) $\log \sin a$	$\cdot \cdot (43^\circ 27' 36'')$	$\cdot \cdot \cdot$	0.162508
$\log \sin B$	$\cdot \cdot \cdot \cdot \cdot \cdot \cdot$	$\cdot \cdot \cdot$	<u>9.852125</u>
$\therefore B = 45^\circ 21' 01''$ , and $B = 134^\circ 38' 59''$ .			

From the first of Napier's Analogies (10), Art. 83, we find,

$$\begin{aligned} \log \cot \frac{1}{2}C &= \log \cos \frac{1}{2}(a+b) + \log \tan \frac{1}{2}(A+B) \\ &\quad + (\text{a. c.}) \log \cos \frac{1}{2}(a-b) - 10. \end{aligned}$$

Taking the first value of  $B$ , we have,

$$\frac{1}{2}(A+B) = 37^\circ 26' 45'';$$

also,

$$\frac{1}{2}(a+b) = 63^\circ 12' 56''; \quad \text{and,} \quad \frac{1}{2}(a-b) = 19^\circ 45' 20''$$

$\log \cos \frac{1}{2}(a+b)$	$\cdot (63^\circ 12' 56'')$	$\cdot$	9.653825
$\log \tan \frac{1}{2}(A+B)$	$\cdot (37^\circ 26' 45'')$	$\cdot$	9.884130
(a. c.) $\log \cos \frac{1}{2}(a-b)$	$\cdot (19^\circ 45' 20'')$	$\cdot$	0.026344
$\log \cot \frac{1}{2}C$	$\cdot \cdot \cdot \cdot \cdot \cdot \cdot$	$\cdot \cdot \cdot$	<u>9.564299</u>

$$\therefore \frac{1}{2}C = 69^\circ 51' 45'', \quad \text{and} \quad C = 139^\circ 43' 30''.$$

The side  $c$  may be found by means of Formula (12), Art. 83, or by means of Formula (2), Art. 78.

Applying logarithms to the proportion,

$$\sin A : \sin C :: \sin a : \sin c, \quad \text{we have,}$$

$$\log \sin c = \log \sin a + \log \sin C + (\text{a. c.}) \log \sin A - 10 ;$$

$\log \sin a$	$(43^\circ 27' 36'')$	9.837492	
$\log \sin C$	$(139^\circ 43' 30'')$	9.810539	
(a. c.) $\log \sin A$	$(29^\circ 32' 29'')$	0.307107	
$\log \sin c$	$\cdot \cdot \cdot \cdot \cdot$	<u>9.955138</u>	$\therefore c = 115^\circ 35' 48''.$

We take the greater value of  $c$ , because the angle  $C$ , being greater than the angle  $B$ , requires that the side  $c$  should be greater than the side  $b$ . By using the second value of  $B$ , we may find, in a similar manner,

$$C = 32^\circ 20' 28'', \quad \text{and} \quad c = 48^\circ 16' 18''.$$

2. Given  $a = 97^\circ 35'$ ,  $b = 27^\circ 08' 22''$ , and  $A = 40^\circ 51' 18''$ , to find  $B$ ,  $C$ , and  $c$ .

$$\text{Ans. } B = 17^\circ 31' 09'', \quad C = 144^\circ 48' 10'', \quad c = 119^\circ 08' 25''.$$

3. Given  $a = 115^\circ 20' 10''$ ,  $b = 57^\circ 30' 06''$ , and  $A = 126^\circ 37' 30''$ , to find  $B$ ,  $C$ , and  $c$ .

$$\text{Ans. } B = 48^\circ 29' 48'', \quad C = 61^\circ 40' 16'', \quad c = 82^\circ 34' 04''.$$

#### CASE II.

*Given two angles and a side opposite one of them.*

86. The solution, in this case, is commenced by finding the side opposite the second given angle, by means of Formula (1), Art. 78. The solution is completed as in Case I.

Since the second side is found by means of its sine, there may be two solutions. To investigate this case, we pass to the polar triangle, by substituting for each part its supplement. In this triangle, there will be given two sides and an angle opposite one; it may therefore be discussed as in the preceding case. When the polar triangle has *two solutions*, *one solution*, or *no solution*, the given triangle will, in like manner, have *two solutions*, *one solution*, or *no solution*.

The conditions may be written out from those of the preceding case, by simply changing *angles* into *sides*, and the reverse; and *greater* into *less*, and the reverse.

Let the given parts be  $A$ ,  $B$ , and  $a$ , and let  $p$  be an arc computed from the equation,



$$\sin p = \sin a \sin B.$$

There will be two cases:  $a$  may be greater than  $90^\circ$ ; or,  $a$  may be less than  $90^\circ$ .

In the first case,

1. When  $A$  is less than  $p$ , and at the same time greater than both  $B$  and  $180^\circ - B$ , there will be two solutions.

2. When  $A$  is less than  $p$ , and intermediate in value between  $B$  and  $180^\circ - B$ ; or, when  $A$  is equal to  $p$ , there will be but one solution.

3. When  $A$  is less than  $p$ , and at the same time less than both  $B$  and  $180^\circ - B$ ; or, when  $A$  is greater than  $p$ , there will be no solution.



In the second case,

4. When  $A$  is greater than  $p$ , and at the same less than both  $B$  and  $180^\circ - B$ , there will be two solutions.

5. When  $A$  is greater than  $p$ , and intermediate in value between  $B$  and  $180^\circ - B$ ; or, when  $A$  is equal to  $p$ , there will be but one solution.

6. When  $A$  is greater than  $p$ , and at the same time greater than both  $B$  and  $180^\circ - B$ ; or, when  $A$  is less than  $p$ , there will be no solution.

#### EXAMPLES.

1. Given  $A = 95^\circ 16'$ ,  $B = 80^\circ 42' 10''$ , and  $a = 57^\circ 38'$ , to find  $a$ ,  $b$ , and  $C$ .

Computing  $p$ , from the formula,

$$\log \sin p = \log \sin B + \log \sin a - 10;$$

we have,  $p = 56^\circ 27' 52''$ .

The smaller value of  $p$  is taken, because  $a$  is less than  $90^\circ$ .

Because  $A > p$ , and intermediate between  $80^\circ 42' 10''$  and  $99^\circ 17' 50''$ , there will, from the fifth condition, be but a single solution.

Applying logarithms to Proportion (1), Art. 78, we have,

$$\log \sin b = \log \sin B + \log \sin a + (\text{a. c.}) \log \sin A - 10;$$

$\log \sin B$	$(80^\circ 42' 10'')$	9.994257	
$\log \sin a$	$(57^\circ 38')$	9.926671	
(a. c.) $\log \sin A$	$(95^\circ 16')$	0.001837	
$\log \sin b$	. . . .	<u>9.922765</u>	$\therefore b = 56^\circ 49' 57''$

We take the smaller value of  $b$ , for the reason that  $A$ , being greater than  $B$ , requires that  $a$  should be greater than  $b$ .

Applying logarithms to Proportion (12), Art. 83, we have,

$$\log \tan \frac{1}{2}c = \log \cos \frac{1}{2}(A + B) + \log \tan \frac{1}{2}(a + b) \\ + (\text{a. c.}) \log \cos \frac{1}{2}(A - B) - 10;$$

we have,

$$\frac{1}{2}(A + B) = 87^{\circ} 59' 05'', \quad \frac{1}{2}(a + b) = 57^{\circ} 13' 58'', \\ \text{and,} \quad \frac{1}{2}(A - B) = 7^{\circ} 16' 55''.$$

log cos $\frac{1}{2}(A + B)$ .	(87° 59' 05'')	.	8.546124
log tan $\frac{1}{2}(a + b)$ .	(57° 13' 58'')	.	10.191352
(a. c.) log cos $\frac{1}{2}(A - B)$ .	( 7° 16' 55'')	.	<u>0.003517</u>
log tan $\frac{1}{2}c$ . . . . .			<u>8.740993</u>

$$\therefore \frac{1}{2}c = 3^{\circ} 09' 09'', \quad \text{and} \quad c = 6^{\circ} 18' 18''.$$

Applying logarithms to the proportion,

$$\sin a : \sin c :: \sin A : \sin C,$$

we have,

$$\log \sin C = \log \sin c + \log \sin A + (\text{a. c.}) \log \sin a - 10;$$

log sin $c$ (6° 18' 18'')	.	9.040685
log sin $A$ (95° 16')	.	9.998163
(a. c.) log sin $a$ (57° 38')	.	<u>0.073329</u>
log sin $C$ . . . . .		<u>9.112177</u> $\therefore C = 7^{\circ} 26' 21''$ .

The smaller value of  $C$  is taken, for the same reason as before.

2. Given  $A = 50^{\circ} 12'$ ,  $B = 58^{\circ} 08'$ , and  $a = 62^{\circ} 42'$ , to find  $b$ ,  $c$ , and  $C$ .

$$b = \begin{cases} 79^{\circ} 12' 10'', \\ 100^{\circ} 47' 50'', \end{cases} \quad c = \begin{cases} 119^{\circ} 03' 26'', \\ 152^{\circ} 14' 18'', \end{cases} \quad C = \begin{cases} 130^{\circ} 54' 28'', \\ 156^{\circ} 15' 06''. \end{cases}$$

## CASE III.

*Given two sides and their included angle.*

87. The remaining angles are found by means of Napier's Analogies, and the remaining side, as in the preceding cases.

## EXAMPLES.

1. Given  $a = 62^\circ 38'$ ,  $b = 10^\circ 13' 19''$ , and  $C = 150^\circ 24' 12''$ , to find  $c$ ,  $A$ , and  $B$ .

Applying logarithms to Proportions (10) and (11), Art. 83, we have,

$$\begin{aligned} \log \tan \frac{1}{2}(A+B) &= \log \cos \frac{1}{2}(a-b) + \log \cot \frac{1}{2}C \\ &\quad + (\text{a. c.}) \log \cos \frac{1}{2}(a+b) - 10; \end{aligned}$$

$$\begin{aligned} \log \tan \frac{1}{2}(A-B) &= \log \sin \frac{1}{2}(a-b) + \log \cot \frac{1}{2}C \\ &\quad + (\text{a. c.}) \log \sin \frac{1}{2}(a+b) - 10; \end{aligned}$$

we have,

$$\frac{1}{2}(a-b) = 26^\circ 12' 20'', \quad \frac{1}{2}C = 75^\circ 12' 06'',$$

and,

$$\frac{1}{2}(a+b) = 36^\circ 25' 39''.$$

log cos $\frac{1}{2}(a-b)$	(26° 12' 20'')	9.952897
log cot $\frac{1}{2}C$	(75° 12' 06'')	9.421901
(a. c.) log cos $\frac{1}{2}(a+b)$	(36° 25' 39'')	0.094415
log tan $\frac{1}{2}(A+B)$	. . . . .	<u>9.469213</u>

$$\therefore \frac{1}{2}(A+B) = 16^\circ 24' 51''.$$

log sin $\frac{1}{2}(a-b)$	(26° 12' 20'')	9.645022
log cot $\frac{1}{2}C$	(75° 12' 06'')	9.421901
(a. c.) log sin $\frac{1}{2}(a+b)$	(36° 25' 39'')	0.226356
log tan $\frac{1}{2}(A-B)$	. . . . .	<u>9.293279</u>

$$\therefore \frac{1}{2}(A-B) = 11^\circ 06' 53''.$$



The greater angle is equal to the half sum plus the half difference, and the less is equal to the half sum minus the half difference. Hence, we have,

$$A = 27^{\circ} 31' 44'', \quad \text{and} \quad B = 5^{\circ} 17' 58''.$$

Applying logarithms to the Proportion (13), Art. 83, we have,

$$\begin{aligned} \log \tan \frac{1}{2}c &= \log \sin \frac{1}{2}(A + B) + \log \tan \frac{1}{2}(a - b) \\ &\quad + (\text{a. c.}) \log \sin \frac{1}{2}(A - B) - 10; \end{aligned}$$

log sin $\frac{1}{2}(A + B)$	·	$(16^{\circ} 24' 51'')$	·	9.451139
log tan $\frac{1}{2}(a - b)$	·	$(26^{\circ} 12' 20'')$	·	9.692125
(a. c.) log sin $\frac{1}{2}(A - B)$	·	$(11^{\circ} 06' 53'')$	·	0.714952
log tan $\frac{1}{2}c$	·	. . . . .	·	<u>9.858216</u>

$$\therefore \frac{1}{2}c = 35^{\circ} 48' 33'', \quad \text{and} \quad c = 71^{\circ} 37' 06''.$$

2. Given  $a = 68^{\circ} 46' 02''$ ,  $b = 37^{\circ} 10'$ , and  $C = 39^{\circ} 23' 23''$ , to find  $c$ ,  $A$ , and  $B$ .

$$\text{Ans. } A = 120^{\circ} 59' 47'', \quad B = 33^{\circ} 45' 03'', \quad c = 43^{\circ} 37' 38''.$$

3. Given  $a = 84^{\circ} 14' 29''$ ,  $b = 44^{\circ} 13' 45''$ , and  $C = 36^{\circ} 45' 28''$ , to find  $A$  and  $B$ .

$$\text{Ans. } A = 130^{\circ} 05' 22'', \quad B = 32^{\circ} 26' 06''.$$

#### CASE IV.

*Given two angles and their included side.*

88. The solution of this case is entirely analogous to that of Case III.

Applying logarithms to Proportions (12) and (13), Art. 83, and to Proportion (11), Art. 83, we have,

$$\log \tan \frac{1}{2}(a + b) = \log \cos \frac{1}{2}(A - B) + \log \tan \frac{1}{2}c \\ + (\text{a. c.}) \log \cos \frac{1}{2}(A + B) - 10 ;$$

$$\log \tan \frac{1}{2}(a - b) = \log \sin \frac{1}{2}(A - B) + \log \tan \frac{1}{2}c \\ + (\text{a. c.}) \log \sin \frac{1}{2}(A + B) - 10 ;$$

$$\log \cot \frac{1}{2}C = \log \sin \frac{1}{2}(a + b) + \log \tan \frac{1}{2}(A - B) \\ + (\text{a. c.}) \log \sin \frac{1}{2}(a - b) - 10 ;$$

The application of these formulas are sufficient for the solution of all cases.

#### EXAMPLES.

1. Given  $A = 81^\circ 38' 20''$ ,  $B = 70^\circ 09' 38''$ , and  $c = 59^\circ 16' 22''$ , to find  $C$ ,  $a$ , and  $b$ .

*Ans.*  $C = 64^\circ 46' 24''$ ,  $a = 70^\circ 04' 17''$ ,  $b = 63^\circ 21' 27''$ .

2. Given  $A = 34^\circ 15' 03''$ ,  $B = 42^\circ 15' 13''$ , and  $c = 76^\circ 35' 36''$ , to find  $C$ ,  $a$ , and  $b$ .

*Ans.*  $C = 121^\circ 36' 12''$ ,  $a = 40^\circ 0' 10''$ ,  $b = 50^\circ 10' 30''$ .

#### CASE V.

*Given the three sides, to find the remaining parts.*

89. The angles may be found by means of Formula (3), Art. 81 ; or, one angle being found by that formula, the other two may be found by means of Napier's Analogies.

#### EXAMPLES.

1. Given  $a = 74^\circ 23'$ ,  $b = 35^\circ 46' 14''$ , and  $c = 100^\circ 39'$ , to find  $A$ ,  $B$ , and  $C$ .

Applying logarithms to Formula (3), Art. 81, we have,

$$\log \cos \frac{1}{2}A = 10 + \frac{1}{2}[\log \sin \frac{1}{2}s + \log \sin (\frac{1}{2}s - a) \\ + (\text{a. c.}) \log \sin b + (\text{a. c.}) \log \sin c - 20];$$

or,

$$\log \cos \frac{1}{2}A = \frac{1}{2}[\log \sin \frac{1}{2}s + \log \sin (\frac{1}{2}s - a) \\ + (\text{a. c.}) \log \sin b + (\text{a. c.}) \log \sin c],$$

we have,

$$\frac{1}{2}s = 105^\circ 24' 07'', \quad \text{and} \quad \frac{1}{2}s - a = 31^\circ 01' 07''.$$

log sin $\frac{1}{2}s$	. . . (105° 24' 07'')	. 9.984116
log sin ( $\frac{1}{2}s - a$ )	. ( 31° 01' 07'')	. 9.712074
(a. c.) log sin $b$	. . . ( 35° 46' 14'')	. 0.233185
(a. c.) log sin $c$	. . . (100° 39')	0.007546
		<hr style="width: 100%; border: 0.5px solid black;"/>
		2)19.936921
log cos $\frac{1}{2}A$	. . . . .	9.968460

$$\therefore \frac{1}{2}A = 21^\circ 34' 23'', \quad \text{and} \quad A = 43^\circ 08' 46''.$$

Using the same formula as before, and substituting  $B$  for  $A$ ,  $b$  for  $a$ , and  $a$  for  $b$ , and recollecting that  $\frac{1}{2}s - b = 69^\circ 37' 53''$ , we have,

log sin $\frac{1}{2}s$	. . . (105° 24' 07'')	. 9.984116
log sin ( $\frac{1}{2}s - b$ )	. ( 69° 37' 53'')	. 9.971958
(a. c.) log sin $a$	. . . (74° 23')	. 0.016336
(a. c.) log sin $c$	. . . (100° 39')	0.007546
		<hr style="width: 100%; border: 0.5px solid black;"/>
		2)19.979956
log cos $\frac{1}{2}B$	. . . . .	9.989978

$$\therefore \frac{1}{2}B = 12^\circ 15' 43'', \quad \text{and} \quad B = 24^\circ 31' 26''.$$

Using the same formula, substituting  $C$  for  $A$ ,  $c$  for  $a$ , and  $a$  for  $c$ , recollecting that  $\frac{1}{2}s - c = 4^\circ 45' 07''$ , we have,



$$\begin{array}{rcll}
\log \sin \frac{1}{2}s & \cdot & \cdot & (105^\circ 24' 07'') \quad \cdot \quad 9.984116 \\
\log \sin (\frac{1}{2}s - c) & \cdot & \cdot & (4^\circ 45' 07'') \quad \cdot \quad 8.918250 \\
(\text{a. c.}) \log \sin \alpha & \cdot & \cdot & (74^\circ 23') \quad \cdot \quad 0.016336 \\
(\text{a. c.}) \log \sin b & \cdot & \cdot & (35^\circ 46' 14'') \quad \cdot \quad 9.233185 \\
& & & \hline
& & & 2) 19.151887 \\
\log \cos \frac{1}{2}C & \cdot & \cdot & \cdot \quad \cdot \quad 9.575943 \\
\therefore \frac{1}{2}C & = & 67^\circ 52' 25'', & \text{and } C = 135^\circ 44' 50''.
\end{array}$$

2. Given  $a = 56^\circ 40'$ ,  $b = 83^\circ 13'$ , and  $c = 114^\circ 30'$ .

*Ans.*  $A = 48^\circ 31' 18''$ ,  $B = 62^\circ 55' 44''$ ,  $C = 125^\circ 18' 56''$ .

#### CASE VI.

*The three angles being given, to find the sides.*

90. The solution in this case is entirely analogous to the preceding one.

Applying logarithms to Formula (2), Art. 82, we have,

$$\begin{aligned}
\log \cos \frac{1}{2}a &= \frac{1}{2}[\log \cos (\frac{1}{2}S - B) + \log \cos (\frac{1}{2}S - C) \\
&\quad + (\text{a. c.}) \log \sin B + (\text{a. c.}) \log \sin C].
\end{aligned}$$

In the same manner as before, we change the letters, to suit each case.

#### EXAMPLES.

1. Given  $A = 48^\circ 30'$ ,  $B = 125^\circ 20'$ , and  $C = 62^\circ 54'$ .

*Ans.*  $a = 56^\circ 39' 30''$ ,  $b = 114^\circ 29' 58''$ ,  $c = 83^\circ 12' 06''$ .

2. Given  $A = 109^\circ 55' 42''$ ,  $B = 116^\circ 38' 33''$ , and  $C = 120^\circ 43' 37''$ , to find  $a$ ,  $b$ , and  $c$ .

*Ans.*  $a = 98^\circ 21' 40''$ ,  $b = 109^\circ 50' 22''$ ,  $c = 115^\circ 13' 28''$ .

# MENSURATION.

---

91. MENSURATION is that branch of Mathematics which treats of the measurement of Geometrical Magnitudes.

92. The measurement of a quantity is the operation of finding how many times it contains another quantity of the same kind, taken as a standard. This standard is called the *unit of measure*.

93. The unit of measure for surfaces is a *square*, one of whose sides is the linear unit. The unit of measure for volumes is a *cube*, one of whose edges is the linear unit.

If the linear unit is *one foot*, the superficial unit is *one square foot*, and the unit of volume is *one cubic foot*. If the linear unit is *one yard*, the superficial unit is *one square yard*, and the unit of volume is *one cubic yard*.

94. In Mensuration, the term *product of two lines*, is used to denote the product obtained by multiplying the number of linear units in one line by the number of linear units in the other. The term *product of three lines*, is used to denote the continued product of the number of linear units in each of the three lines.

Thus, when we say that the area of a parallelogram is equal to the product of its base and altitude, we mean that the number of superficial units in the parallelogram is equal to the number of linear units in the base, multiplied by the number of linear units in the altitude. In like manner, the

number of units of volume, in a rectangular parallelopipedon, is equal to the number\* of superficial units in its base multiplied by the number of linear units in its altitude, and so on.

### MENSURATION OF PLANE FIGURES.

*To find the area of a parallelogram.*

95. From the principle demonstrated in Book IV., Prop. V., we have the following

#### RULE.

*Multiply the base by the altitude; the product will be the area required.*

#### EXAMPLES.

1. Find the area of a parallelogram, whose base is 12.25, and whose altitude is 8.5. *Ans.* 104.125.

2. What is the area of a square, whose side is 204.3 feet? *Ans.* 41738.49 sq. ft.

3. How many square yards are there in a rectangle, whose base is 66.3 feet, and altitude 33.3 feet? *Ans.* 245.31 sq. yd.

4. What is the area of a rectangular board, whose length is  $12\frac{1}{2}$  feet, and breadth 9 inches? *Ans.*  $9\frac{3}{8}$  sq. ft.

5. What is the number of square yards in a parallelogram, whose base is 37 feet, and altitude 5 feet 3 inches? *Ans.*  $21\frac{7}{17}$ .

*To find the area of a plane triangle.*

96. *First Case.* When the base and altitude are given.



From the principle demonstrated in Book X., Prop. VI., we may write the following

## RULE.

*Multiply the base by half the altitude ; the product will be the area required.*

## EXAMPLES.

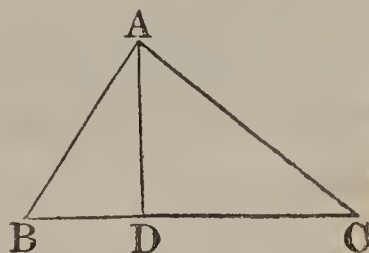
1. Find the area of a triangle, whose base is 625, and altitude 520 feet. *Ans.* 162500 sq. ft.

2. Find the area of a triangle, in square yards, whose base is 40, and altitude 30 feet. *Ans.*  $66\frac{2}{3}$ .

3. Find the area of a triangle, in square yards, whose base is 49, and altitude  $25\frac{1}{4}$  feet. *Ans.* 68.7361.

*Second Case.* When two sides and their included angle are given.

Let  $ABC$  represent a plane triangle, in which the side  $AB = c$ ,  $BC = a$ , and the angle  $B$ , are given. From  $A$  draw  $AD$  perpendicular to  $BC$  ; this will be the altitude of the triangle. From Formula (1), Art. 37, Plane Trigonometry, we have,



$$AD = c \sin B.$$

Denoting the area of the triangle by  $Q$ , and applying the rule last given, we have,

$$Q = \frac{ac \sin B}{2} ; \quad \text{or,} \quad 2Q = ac \sin B.$$

Substituting for  $\sin B$ ,  $\frac{\sin B}{R}$  (Trig., Art. 30), and applying logarithms, we have,

$$\log (2Q) = \log a + \log c + \log \sin B - 10 ;$$

hence, we may write the following

RULE.

*Add together the logarithms of the two sides and the logarithmic sine of their included angle ; from this sum subtract 10 ; the remainder will be the logarithm of double the area of the triangle. Find, from the table, the number answering to this logarithm, and divide it by 2 ; the quotient will be the required area.*

EXAMPLES.

1. What is the area of a triangle, in which two sides  $a$  and  $b$ , are respectively equal to 125.81, and 57.65, and whose included angle  $C$ , is  $57^\circ 25'$ ?

*Ans.*  $2Q = 6111.4$ , and  $Q = 3055.7$  *Ans.*

2. What is the area of a triangle, whose sides are 30 and 40, and their included angle  $28^\circ 57'$ ? *Ans.* 290.427.

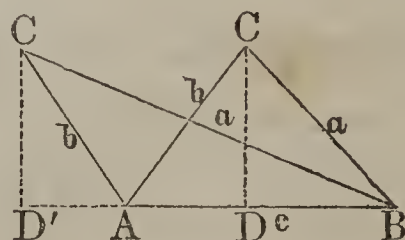
3. What is the number of square yards in a triangle, of which the sides are 25 feet and 21.25 feet, and their included angle  $45^\circ$ ? *Ans.* 20.8694.

LEMMA.

*To find half an angle, when the three sides of a plane triangle are given.*

97. Let  $ABC$  be a plane triangle, the angles and sides being denoted as in the figure.

We have (B. IV., P. XII., XIII.),



$$a^2 = b^2 + c^2 \mp 2c \cdot AD. \quad \dots \quad (1.)$$

When the angle  $A$  is acute, we have (Art. 37),

$$AD = b \cos A; \quad \text{when obtuse,} \quad AD' = b \cos CAD'.$$

But as  $CAD'$  is the supplement of the obtuse angle  $A$ ,

$$\cos CAD' = -\cos A, \quad \text{and} \quad AD' = -b \cos A.$$

Either of these values, being substituted for  $AD$ , in (1), gives,

$$a^2 = b^2 + c^2 - 2bc \cos A;$$

whence,

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \dots \dots \dots (2.)$$

If we add 1 to both members, and recollect that  $1 + \cos A = 2 \cos^2 \frac{1}{2}A$  (Art. 66), Equation (4), we have,

$$\begin{aligned} 2 \cos^2 \frac{1}{2}A &= \frac{2bc + b^2 + c^2 - a^2}{2bc} \\ &= \frac{(b+c)^2 - a^2}{2bc} = \frac{(b+c+a)(b+c-a)}{2bc}; \end{aligned}$$

or,

$$\cos^2 \frac{1}{2}A = \frac{(b+c+a)(b+c-a)}{4bc} \dots \dots \dots (3.)$$

If we put  $b+c+a = s$ , we have,

$$\frac{b+c+a}{2} = \frac{1}{2}s, \quad \text{and,} \quad \frac{b+c-a}{2} = \frac{1}{2}s - a;$$

Substituting in (3), and extracting the square root,

$$\cos \frac{1}{2}A = \sqrt{\frac{\frac{1}{2}s (\frac{1}{2}s - a)}{bc}}, \dots \dots \dots (4.)$$

the plus sign, only, being used, since  $\frac{1}{2}A < 90^\circ$ ; hence,

*The cosine of half of either angle of a plane triangle, is equal to the square root of half the sum of the three sides, into half that sum minus the side opposite the angle, divided by the rectangle of the adjacent sides.*

By applying logarithms, we have,  $\log \cos \frac{1}{2}A =$

$$\frac{1}{2} [\log \frac{1}{2}s + \log (\frac{1}{2}s - a) + (\text{a. c.}) \log b + (\text{a. c.}) \log c]. \quad (\text{A.})$$



If we subtract both members of Equation (2), from 1, and recollect that  $1 - \cos A = 2 \sin^2 \frac{1}{2}A$  (Art. 37), we have,

$$\begin{aligned} 2 \sin^2 \frac{1}{2}A &= \frac{2bc - b^2 - c^2 + a^2}{2bc} \\ &= \frac{a^2 - (b - c)^2}{2bc} = \frac{(a + b - c)(a - b + c)}{2bc} \quad (5.) \end{aligned}$$

Placing, as before,  $a + b + c = s$ , we have,

$$\frac{a + b - c}{2} = \frac{1}{2}s - c, \quad \text{and,} \quad \frac{a - b + c}{2bc} = \frac{1}{2}s - b.$$

Substituting in (5), and reducing, we have,

$$\sin \frac{1}{2}A = \sqrt{\frac{(\frac{1}{2}s - b)(\frac{1}{2}s - c)}{bc}} \quad (6.)$$

hence,

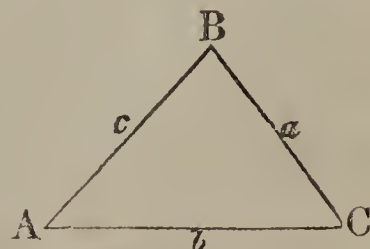
*The sine of half an angle of a plane triangle, is equal to the square root of half the sum of the three sides, minus one of the adjacent sides, into the half sum minus the other adjacent side, divided by the rectangle of the adjacent sides.*

Applying logarithms, we have,

$$\begin{aligned} \log \sin \frac{1}{2}A &= \frac{1}{2} [\log (\frac{1}{2}s - b) + \log (\frac{1}{2}s - c) \\ &\quad + (\text{a. c.}) \log b + (\text{a. c.}) \log c]. \quad (B.) \end{aligned}$$

*Third Case.* To find the area of a triangle, when the three sides are given.

Let  $ABC$  represent a triangle whose sides  $a$ ,  $b$ , and  $c$  are given. From the principle demonstrated in the last case, we have,



$$Q = \frac{1}{2}bc \sin A.$$

But, from Formula ( $\Delta'$ ), Trig., Art. 66, we have,

$$\sin A = 2 \sin \frac{1}{2}A \cos \frac{1}{2}A ;$$

whence,

$$Q = bc \sin \frac{1}{2}A \cos \frac{1}{2}A.$$

Substituting for  $\sin \frac{1}{2}A$  and  $\cos \frac{1}{2}A$ , their values, taken from Lemma, and reducing, we have,

$$Q = \sqrt{\frac{1}{2}s (\frac{1}{2}s - a) (\frac{1}{2}s - b) (\frac{1}{2}s - c)} ;$$

hence, we may write the following

#### RULE.

*Find half the sum of the three sides, and from it subtract each side separately. Find the continued product of the half sum and the three remainders, and extract its square root; the result will be the area required.*

It is generally more convenient to employ logarithms; for this purpose, applying logarithms to the last equation, we have,

$$\log Q = \frac{1}{2}[\log \frac{1}{2}s + \log (\frac{1}{2}s - a) + \log (\frac{1}{2}s - b) + \log (\frac{1}{2}s - c)]$$

hence, we have the following

#### RULE.

*Find the half sum and the three remainders as before, then find the half sum of their logarithms; the number corresponding to the resulting logarithm will be the area required.*

#### EXAMPLES.

1. Find the area of a triangle, whose sides are 20, 30, and 40.

We have,  $\frac{1}{2}s = 45$ ,  $\frac{1}{2}s - a = 25$ ,  $\frac{1}{2}s - b = 15$ ,  $\frac{1}{2}s - c = 5$ .  
By the first rule,

$$Q = \sqrt{45 \times 25 \times 15 \times 5} = 290.4737 \text{ Ans.}$$

By the second rule,

$$\begin{array}{rcl}
 \log \frac{1}{2}s & . & . & . & . & (45) & . & . & . & . & 1.653213 \\
 \log (\frac{1}{2}s - a) & . & . & . & . & (25) & . & . & . & . & 1.397940 \\
 \log (\frac{1}{2}s - b) & . & . & . & . & (15) & . & . & . & . & 1.176091 \\
 \log (\frac{1}{2}s - c) & . & . & . & . & (5) & . & . & . & . & 0.698970 \\
 & & & & & & & & & & \hline
 & & & & & & & & & & 2)4.926214 \\
 \log Q & . & . & . & . & . & . & . & . & . & 2.463107
 \end{array}$$

$$\therefore Q = 290.4737 \text{ Ans.}$$

2. How many square yards are there in a triangle, whose sides are 30, 40, and 50 feet? Ans.  $66\frac{2}{3}$ .

*To find the area of a trapezoid.*

98. From the principle demonstrated in Book IV., Prop. VII., we may write the following

#### RULE.

*Find half the sum of the parallel sides, and multiply it by the altitude; the product will be the area required.*

#### EXAMPLES.

1. In a trapezoid the parallel sides are 750 and 1225, and the perpendicular distance between them is 1540; what is the area? Ans. 1520750.

2. How many square feet are contained in a plank, whose length is 12 feet 6 inches, the breadth at the greater end 15 inches, and at the less end 11 inches? Ans.  $13\frac{1}{2}$ .

3. How many square yards are there in a trapezoid, whose parallel sides are 240 feet, 320 feet, and altitude 66 feet? Ans.  $2053\frac{1}{3}$  sq. yd.

*To find the area of any quadrilateral.*

99. From what precedes, we deduce the following



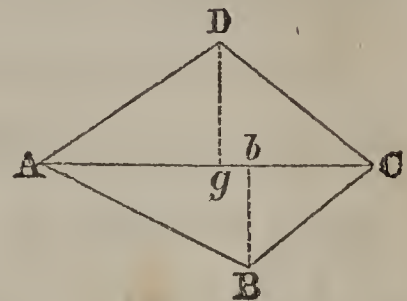
RULE.

Join the vertices of two opposite angles by a diagonal; from each of the other vertices let fall perpendiculars upon this diagonal; multiply the diagonal by half of the sum of the perpendiculars, and the product will be the area required.

EXAMPLES.

1. What is the area of the quadrilateral  $ABCD$ , the diagonal  $AC$  being 42, and the perpendiculars  $Dg$ ,  $Bb$ , equal to 18 and 16 feet?

*Ans.* 714 sq. ft.



2. How many square yards of paving are there in the quadrilateral, whose diagonal is 65 feet, and the two perpendiculars let fall on it 28 and  $33\frac{1}{2}$  feet? *Ans.*  $222\frac{1}{2}$ .

*To find the area of any polygon.*

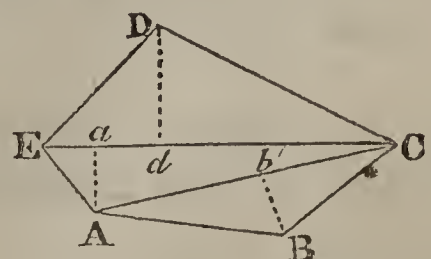
100. From what precedes, we have the following

RULE.

Draw diagonals dividing the proposed polygon into trapezoids and triangles: then find the areas of these figures separately, and add them together for the area of the whole polygon.

EXAMPLE.

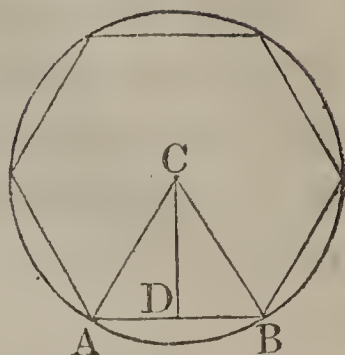
1. Let it be required to determine the area of the polygon  $ABCDE$ , having five sides.



Let us suppose that we have measured the diagonals and perpendiculars, and found  $AC = 36.21$ ,  $EC = 39.11$ ,  $Bb = 4$ ,  $Dd = 7.26$ ,  $Aa = 4.18$ : required the area. *Ans.* 296.1292.

*To find the area of a regular polygon.*

101. Let  $AB$ , denoted by  $s$ , represent one side of a regular polygon, whose centre is  $C$ . Draw  $CA$  and  $CB$ , and from  $C$  draw  $CD$  perpendicular to  $AB$ . Then will  $CD$  be the apothem, and we shall have  $AD = BD$ .



Denote the number of sides of the polygon by  $n$ ; then will the angle  $ACB$ , at the centre, be equal to  $\frac{360^\circ}{n}$ , (B. V., Page 138, D. 2), and the angle  $ACD$ , which is half of  $ACB$ , will be equal to  $\frac{180^\circ}{n}$ .

In the right-angled triangle  $ADC$ , we shall have, Formula (3), Art. 37, Trig.,

$$CD = \frac{1}{2}s \tan CAD.$$

But  $CAD$ , being the complement of  $ACD$ , we have,  
 $\tan CAD = \cot ACD$ ;

hence, 
$$CD = \frac{1}{2}s \cot \frac{180^\circ}{n},$$

a formula by means of which the apothem may be computed.

But the area is equal to the perimeter multiplied by half the apothem (Book V., Prop. VIII.): hence the following

#### RULE.

*Find the apothem, by the preceding formula; multiply the perimeter by half the apothem; the product will be the area required.*

#### EXAMPLES.

1. What is the area of a regular hexagon, each of whose sides is 20? We have,

$$CD = 10 \times \cot 30^\circ; \quad \text{or,} \quad \log CD = \log 10 + \log \cot 30^\circ - 10.$$

$$\log \frac{1}{2}s \quad . \quad . \quad . \quad (10) \quad . \quad 1.000000$$

$$\log \cot \frac{180^\circ}{n} \quad . \quad (30^\circ) \quad . \quad 10.238561$$

$$\log CD \quad . \quad . \quad . \quad . \quad \underline{1.238561} \quad \therefore CD = 17.3205.$$

The perimeter is equal to 120 : hence, denoting the area by  $Q$ ,

$$Q = \frac{120 \times 17.3205}{2} = 1039.23 \text{ Ans.}$$

2. What is the area of an octagon, one of whose sides is 20 ? Ans. 1931.36886.

The areas of some of the most important of the regular polygons have been computed by the preceding method, on the supposition that each side is equal to 1, and the results are given in the following

TABLE.

NAMES.	SIDES.	AREAS.	NAMES.	SIDES.	AREAS.
Triangle, . .	3 . .	0.4330127	Octagon, . .	8 . .	4.8284271
Square, . .	4 . .	1.0000000	Nonagon, . .	9 . .	6.1818242
Pentagon, . .	5 . .	1.7204774	Decagon, . .	10 . .	7.6942088
Hexagon . .	6 . .	2.5980762	Undecagon, .	11 . .	9.3656399
Heptagon . .	7 . .	3.6339124	Dodecagon, .	12 . .	11.1961524

The areas of similar polygons are to each other as the squares of their homologous sides (Book IV., Prop. XXVII.).

Denoting the area of a regular polygon whose side is  $s$ , by  $Q$ , and that of a similar polygon whose side is 1, by  $T$ , the tabular area, we have,

$$Q : T :: s^2 : 1^2 ; \therefore Q = Ts^2 ;$$

hence, the following RULE.

*Multiply the corresponding tabular area by the square of the given side ; the product will be the area required.*

EXAMPLES.

1. What is the area of a regular hexagon, each of whose sides is 20 ?

We have,  $T = 2.5980762$ , and  $s^2 = 400$  : hence,

$$Q = 2.5980762 \times 400 = 1039.23048 \text{ Ans.}$$



2. Find the area of a pentagon, whose side is 25.

*Ans.* 1075.298375.

3. Find the area of a decagon, whose side is 20.

*Ans.* 3077.68352.

*To find the circumference of a circle, when the diameter is given.*

102. From the principle demonstrated in Book V., Prop. XVI., we may write the following

RULE.

*Multiply the given diameter by 3.1416 ; the product will be the circumference required.*

EXAMPLES.

1. What is the circumference of a circle, whose diameter is 25 ?

*Ans.* 78.54.

2. If the diameter of the earth is 7921 miles, what is the circumference ?

*Ans.* 24884.6136.

*To find the diameter of a circle, when the circumference is given.*

103. From the preceding case, we may write the following

RULE.

*Divide the given circumference by 3.1416 ; the quotient will be the diameter required.*

EXAMPLES.

1. What is the diameter of a circle, whose circumference is 11652.1944 ?

*Ans.* 3709.

2. What is the diameter of a circle, whose circumference is 6850 ?

*Ans.* 2180.41.

*To find the length of an arc containing any number of degrees.*

104. The length of an arc of  $1^\circ$ , in a circle whose diameter is 1, is equal to the circumference, or 3.1416 divided by 360; that is, it is equal to 0.0087266: hence, the length of an arc of  $n$  degrees, will be,  $n \times 0.0087266$ . To find the length of an arc containing  $n$  degrees, when the diameter is  $d$ , we employ the principle demonstrated in Book V., Prop. XIII., C. 2: hence, we may write the following

## RULE.

*Multiply the number of degrees in the arc by .0087266, and the product by the diameter of the circle; the result will be the length required.*

## EXAMPLES.

1. What is the length of an arc of 30 degrees, the diameter being 18 feet? *Ans.* 4.712364 ft.
2. What is the length of an arc of  $12^\circ 10'$ , or  $12\frac{1}{6}^\circ$ , the diameter being 20 feet? *Ans.* 2.123472 ft.

*To find the area of a circle.*

105. From the principle demonstrated in Book V., Prop. XV., we may write the following

## RULE.

*Multiply the square of the radius by 3.1416; the product will be the area required.*

## EXAMPLES.

1. Find the area of a circle, whose diameter is 10, and circumference 31.416. *Ans.* 78.54.
2. How many square yards in a circle whose diameter is  $3\frac{1}{2}$  feet? *Ans.* 1.069016.
3. What is the area of a circle whose circumference is 12 feet? *Ans.* 11.4595.

*To find the area of a circular sector.*

106. From the principle demonstrated in Book V., Prop. XIV., C. 1 and 2, we may write the following

RULE.

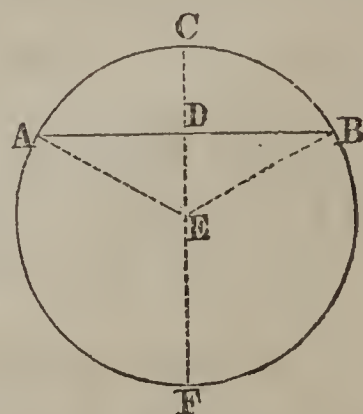
- I. *Multiply half the arc by the radius ; or,*
- II. *Find the area of the whole circle, by the last rule ; then write the proportion, as 360 is to the number of degrees in the sector, so is the area of the circle to the area of the sector.*

EXAMPLES.

1. Find the area of a circular sector, whose arc contains  $18^\circ$ , the diameter of the circle being 3 feet. 0.35343 sq. ft.
2. Find the area of a sector, whose arc is 20 feet, the radius being 10. Ans. 100.
3. Required the area of a sector, whose arc is  $147^\circ 29'$ , and radius 25 feet. Ans. 804.3986 sq. ft.

*To find the area of a circular segment.*

107. Let  $AB$  represent the chord corresponding to the two segments  $ACB$  and  $AFB$ . Draw  $AE$  and  $BE$ . The segment  $ACB$  is equal to the sector  $EACB$ , minus the triangle  $AEB$ . The segment  $AFB$  is equal to the sector  $EAFB$ , plus the triangle  $AEB$ . Hence, we have the following



RULE.

*Find the area of the corresponding sector, and also of the triangle formed by the chord of the segment and the two extreme radii of the sector ; subtract the latter from the former when the segment is less than a semicircle, and take their sum when the segment is greater than a semicircle ; the result will be the area required.*



## EXAMPLES.

1. Find the area of a segment, whose chord is 12 and the radius 10.

Solving the triangle  $AEB$ , we find the angle  $AEB$  is equal to  $73^\circ 44'$ , the area of the sector  $EACB$  equal to 64.35, and the area of the triangle  $AEB$  equal to 48; hence, the segment  $ACB$  is equal to 16.35 *Ans.*

2. Find the area of a segment, whose height is 18, the diameter of the circle being 50. *Ans.* 636.4834.

3. Required the area of a segment, whose chord is 16, the diameter being 20. *Ans.* 44.764.

*To find the area of a circular ring contained between the circumferences of two concentric circles.*

108. Let  $R$  and  $r$  denote the radii of the two circles,  $R$  being greater than  $r$ . The area of the outer circle is  $R^2 \times 3.1416$ , and that of the inner circle is  $r^2 \times 3.1416$ ; hence, the area of the ring is equal to  $(R^2 - r^2) \times 3.1416$ . Hence, the following

## RULE.

*Find the difference of the squares of the radii of the two circles, and multiply it by 3.1416; the product will be the area required.*

## EXAMPLES.

1. The diameters of two concentric circles being 10 and 6, required the area of the ring contained between their circumferences. *Ans.* 50.2656.

2. What is the area of the ring, when the diameters of the circles are 10 and 20? *Ans.* 235.62.

## MENSURATION OF BROKEN AND CURVED SURFACES.

*To find the area of the entire surface of a right prism.*

109. From the principle demonstrated in Book VII., Prop. I., we may write the following

## RULE.

*Multiply the perimeter of the base by the altitude, the product will be the area of the convex surface; to this add the areas of the two bases; the result will be the area required.*

## EXAMPLES.

1. Find the surface of a cube, the length of each side being 20 feet. *Ans.* 2400 sq. ft.

2. Find the whole surface of a triangular prism, whose base is an equilateral triangle, having each of its sides equal to 18 inches, and altitude 20 feet. *Ans.* 91.949 sq. ft.

*To find the area of the entire surface of a right pyramid.*

110. From the principle demonstrated in Book VII., Prop. IV., we may write the following

## RULE.

*Multiply the perimeter of the base by half the slant height; the product will be the area of the convex surface; to this add the area of the base; the result will be the area required.*

## EXAMPLES.

1. Find the convex surface of a right triangular pyramid, the slant height being 20 feet, and each side of the base 3 feet. *Ans.* 90 sq. ft.

2. What is the entire surface of a right pyramid, whose slant height is 15 feet, and the base a pentagon, of which each side is 25 feet? *Ans.* 2012.798 sq. ft.

*To find the area of the convex surface of a frustum of a right pyramid.*

III. From the principle demonstrated in Book XII., Prop. IV., C., we may write the following

R U L E .

*Multiply the half sum of the perimeters of the two bases by the slant height ; the product will be the area required.*

E X A M P L E S .

1. How many square feet are there in the convex surface of the frustum of a square pyramid, whose slant height is 10 feet, each side of the lower base 3 feet 4 inches, and each side of the upper base 2 feet 2 inches? *Ans.* 110 sq. ft.

2. What is the convex surface of the frustum of a heptagonal pyramid, whose slant height is 55 feet, each side of the lower base 8 feet, and each side of the upper base 4 feet? *Ans.* 2310 sq. ft.

112. Since a cylinder may be regarded as a prism whose base has an infinite number of sides, and a cone as a pyramid whose base has an infinite number of sides, the rules just given, may be applied to find the areas of the surfaces of right cylinders, cones, and frustums of cones, by simply changing the term *perimeter*, to circumference.

E X A M P L E S .

1. What is the convex surface of a cylinder, the diameter of whose base is 20, and whose altitude 50? *Ans.* 3141.6.

2. What is the entire surface of a cylinder, the altitude being 20, and diameter of the base 2 feet? 131.9472 sq. ft.

3. Required the convex surface of a cone, whose slant height is 50 feet, and the diameter of its base  $8\frac{1}{2}$  feet.

*Ans.* 667.59 sq. ft.



4. Required the entire surface of a cone, whose slant height is 36, and the diameter of its base 18 feet.

*Ans.* 1272.348 sq. ft.

5. Find the convex surface of the frustum of a cone, the slant height of the frustum being  $12\frac{1}{2}$  feet, and the circumferences of the bases 8.4 feet and 6 feet. *Ans.* 90 sq. ft.

6. Find the entire surface of the frustum of a cone, the slant height being 16 feet, and the radii of the bases 3 feet, and 2 feet. *Ans.* 292.1688 sq. ft.

*To find the area of the surface of a sphere.*

113. From the principle demonstrated in Book VIII., Prop. X., C. 1, we may write the following

#### RULE.

*Find the area of one of its great circles, and multiply it by 4; the product will be the area required.*

#### EXAMPLES.

1. What is the area of the surface of a sphere, whose radius is 16? *Ans.* 3216.9984.

2. What is the area of the surface of a sphere, whose radius is 27.25 *Ans.* 9331.3374.

*To find the area of a zone.*

114. From the principle demonstrated in Book VIII., Prop. X., C. 2, we may write the following

#### RULE.

*Find the circumference of a great circle of the sphere, and multiply it by the altitude of the zone; the product will be the area required.*

## EXAMPLES.

1. The diameter of a sphere being 42 inches, what is the area of the surface of a zone whose altitude is 9 inches.

*Ans.* 1187.5248 sq. in.

2. If the diameter of a sphere is  $12\frac{1}{2}$  feet, what will be the surface of a zone whose altitude is 2 feet? 78.54 sq. ft.

*To find the area of a spherical polygon.*

115. From the principle demonstrated in Book IX., Prop. XIX., we may write the following

## RULE.

*From the sum of the angles of the polygon, subtract  $180^\circ$  taken as many times as the polygon has sides, less two, and divide the remainder by  $90^\circ$ ; the quotient will be the spherical excess. Find the area of a great circle of the sphere, and divide it by 2; the quotient will be the area of a tri-rectangular triangle. Multiply the area of the tri-rectangular triangle by the spherical excess, and the product will be the area required.*

This rule applies to the spherical triangle, as well as to any other spherical polygon.

## EXAMPLES.

1. Required the area of a triangle described on a sphere, whose diameter is 30 feet, the angles being  $140^\circ$ ,  $92^\circ$ , and  $68^\circ$ . *Ans.* 471.24 sq. ft.

2. What is the area of a polygon of seven sides, described on a sphere whose diameter is 17 feet, the sum of the angles being  $1080^\circ$ ? *Ans.* 226.98.

3. What is the area of a regular polygon of eight sides, described on a sphere whose diameter is 30 yards, each angle of the polygon being  $140^\circ$ ? *Ans.* 157.08 sq. yds.

## MENSURATION OF VOLUMES.

*To find the volume of a prism.*

116. From the principle demonstrated in Book VII., Prop. XIV., we may write the following

## RULE.

*Multiply the area of the base by the altitude ; the product will be the volume required.*

## EXAMPLES.

1. What is the volume of a cube, whose side is 24 inches?

*Ans.* 13824 cu. in.

2. How many cubic feet in a block of marble, of which the length is 3 feet 2 inches, breadth 2 feet 8 inches, and height or thickness 2 feet 6 inches?

*Ans.*  $21\frac{1}{9}$  cu. ft.

3. Required the volume of a triangular prism, whose height is 10 feet, and the three sides of its triangular base 3, 4, and 5 feet.

*Ans.* 60.

*To find the volume of a pyramid.*

117. From the principle demonstrated in Book VII., Prop. XVII., we may write the following

## RULE.

*Multiply the area of the base by one-third of the altitude ; the product will be the volume required.*

## EXAMPLES.

1. Required the volume of a square pyramid, each side of its base being 30, and the altitude 25.

*Ans.* 7500.

2. Find the volume of a triangular pyramid, whose altitude is 30, and each side of the base 3 feet.

38.9711 cu. ft.



3. What is the volume of a pentagonal pyramid, its altitude being 12 feet, and each side of its base 2 feet.

*Ans.* 27.5276 cu. ft.

4. What is the volume of an hexagonal pyramid, whose altitude is 6.4 feet, and each side of its base 6 inches?

*Ans.* 1.38564 cu. ft.

*To find the volume of a frustum of a pyramid.*

118. From the principle demonstrated in Book VII., Prop., XVIII., C., we may write the following

#### RULE.

*Find the sum of the upper base, the lower base, and a mean proportional between them; multiply the result by one-third of the altitude; the product will be the volume required.*

#### EXAMPLES.

1. Find the number of cubic feet in a piece of timber, whose bases are squares, each side of the lower base being 15 inches, and each side of the upper base 6 inches, the altitude being 24 feet. *Ans.* 19.5.

2. Required the volume of a pentagonal frustum, whose altitude is 5 feet, each side of the lower base 18 inches, and each side of the upper base 6 inches. *Ans.* 9.31925 cu. ft.

119. Since cylinders and cones are limiting cases of prisms and pyramids, the three preceding rules are equally applicable to them.

#### EXAMPLES.

1. Required the volume of a cylinder whose altitude is 12 feet, and the diameter of its base 15 feet.

*Ans.* 2120.58 cu. ft.

2. Required the volume of a cylinder whose altitude is 20 feet, and the circumference of whose base is 5 feet 6 inches. *Ans.* 48.144 cu. ft.

3. Required the volume of a cone whose altitude is 27 feet, and the diameter of the base 10 feet.

*Ans.* 706.86 cu. ft.

4. Required the volume of a cone whose altitude is  $10\frac{1}{2}$  feet, and the circumference of its base 9 feet.

*Ans.* 22.56 cu. ft.

5. Find the volume of the frustum of a cone, the altitude being 18, the diameter of the lower base 8, and that of the upper base 4.

*Ans.* 527.7888.

6. What is the volume of the frustum of a cone, the altitude being 25, the circumference of the lower base 20, and that of the upper base 10?

*Ans.* 464.216.

7. If a cask, which is composed of two equal conic frustums joined together at their larger bases, have its bung diameter 28 inches, the head diameter 20 inches, and the length 40 inches, how many gallons of wine will it contain, there being 231 cubic inches in a gallon?

*Ans.* 79.0613.

*To find the volume of a sphere.*

120. From the principle demonstrated in Book VIII., Prop. XIV., we may write the following

#### RULE.

*Cube the diameter of the sphere, and multiply the result by  $\frac{1}{6}\pi$ , that is, by 0.5236; the product will be the volume required.*

#### EXAMPLES.

1. What is the volume of a sphere, whose diameter is 12?

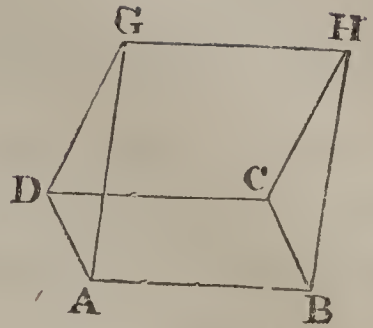
*Ans.* 904.7808.

2. What is the volume of the earth, if the mean diameter be taken equal to 7918.7 miles.

*Ans.* 259992792083 cu. miles.

*To find the volume of a wedge.*

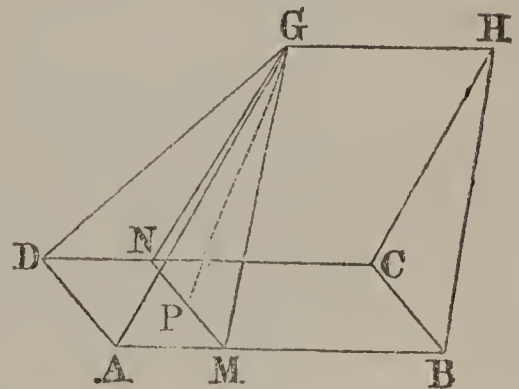
121. A WEDGE is a volume bounded by a rectangle  $ABCD$ , called the *back*, two trapezoids  $ABHG$ ,  $DCHG$ , called *faces*, and two triangles  $ADG$ ,  $CBH$ , called *ends*. The line  $GH$ , in which the faces meet, is called the *edge*. The two faces are equally inclined to the back, and so also are the two ends.



There are three cases: 1st, When the length of the edge is equal to the length of the back; 2d, When it is less; and 3d, When it is greater.

In the first case, the wedge is a right prism, whose base is the triangle  $ADG$ , and altitude  $GH$  or  $AB$ : hence, its volume is equal to  $ADG$  multiplied by  $AB$ .

In the second case, through  $H$ , the middle point of the edge, pass a plane  $HCB$  perpendicular to the back and intersecting it in the line  $BC$  parallel to  $AD$ . This plane will divide the wedge into two parts, one of which is represented by the figure.



Through  $G$ , draw the plane  $GNM$  parallel to  $HCB$ , and it will divide the part of the wedge represented by the figure into the right triangular prism  $GNM-B$ , and the quadrangular pyramid  $ADNM-G$ . Draw  $GP$  perpendicular to  $NM$ : it will also be perpendicular to the back of the wedge (B. VI., P. XVII.), and hence, will be equal to the altitude of the wedge.

Denote  $AB$  by  $L$ , the breadth  $AD$  by  $b$ , the edge  $GH$  by  $l$ , the altitude by  $h$ , and the volume by  $V$ ; then,

$AM = L - l$ ,  $MB = GH = l$ , and area  $NGM = \frac{1}{2}bh$ : then

Prism  $= \frac{1}{2}bhl$ ; Pyramid  $= b(L - l)\frac{1}{3}h = \frac{1}{3}bh(L - l)$ , and

$V = \frac{1}{2}bhl + \frac{1}{3}bh(L - l) = \frac{1}{2}bhl + \frac{1}{3}bhL - \frac{1}{3}bhl = \frac{1}{6}bh(l + 2L)$ .

We can find a similar expression for the remaining part of the wedge, and by adding, the factor within the parenthesis becomes the entire length of the edge plus twice the length of the back.



In the third case,  $l$  is greater than  $L$ , and denotes the altitude of the prism; the volume of each part is equal to the difference of the prism and pyramid, and is of the same form as before. Hence, the following

**RULE.**—*Add twice the length of the back to the length of the edge; multiply the sum by the breadth of the back, and that result by one-sixth of the altitude; the final product will be the volume required.*

#### EXAMPLES.

1. If the back of a wedge is 40 by 20 feet, the edge 35 feet, and the altitude 10 feet, what is the volume?

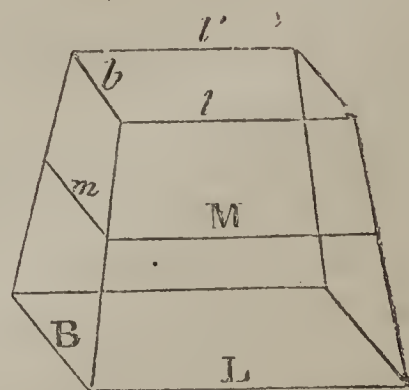
*Ans.* 3833.33 cu.ft.

2. What is the volume of a wedge, whose back is 18 feet by 9, edge 20 feet, and altitude 6 feet? 504 cu. ft.

*To find the volume of a prismoid.*

**122.** A PRISMOID is a frustum of a wedge.

Let  $L$  and  $B$  denote the length and breadth of the lower base,  $l$  and  $b$  the length and breadth of the upper base,  $M$  and  $m$  the length and breadth of the section equidistant from the bases, and  $h$  the altitude of the prismoid.



Through the edges  $L$  and  $l'$ , let a plane be passed, and it will divide the prismoid into two wedges, having for bases, the bases of the prismoid, and for edges the lines  $L$  and  $l'$ .

The volume of the prismoid, denoted by  $V$ , will be equal to the sum of the volumes of the two wedges; hence,

$$V = \frac{1}{6}Bh(l + 2L) + \frac{1}{6}bh(L + 2l);$$

or,

$$V = \frac{1}{6}h(2BL + 2bl + Bl + bL);$$

which may be written under the form,

$$V = \frac{1}{6}h[(BL + bl + Bl + bL) + BL + bl]. \quad \cdot \cdot \quad (\text{A.})$$

Because the auxiliary section is midway between the bases, we have,

$$2M = L + l, \quad \text{and} \quad 2m = B + b;$$

hence,

$$4Mm = (L + l)(B + b) = BL + bl + Bl + bL.$$

Substituting in (A), we have,

$$V = \frac{1}{6}h(BL + bl + 4Mm).$$

But  $BL$  is the area of the lower base, or lower section,  $bl$  is the area of the upper base, or upper section, and  $Mm$  is the area of the middle section; hence, the following

#### RULE.

*To find the volume of a prismoid, find the sum of the areas of the extreme sections and four times the middle section; multiply the result by one-sixth of the distance between the extreme sections; the result will be the volume required.*

This rule is used in computing volumes of earth-work in railroad cutting and embankment, and is of very extensive application. It may be shown that the same rule holds for every one of the volumes heretofore discussed in this work. Thus, in a pyramid, we may regard the base as one extreme section, and the vertex (whose area is 0), as the other extreme; their sum is equal to the area of the base. The area of a section midway between them is equal to one-fourth of the base: hence, four times the middle section is equal to the base. Multiplying the sum of these by one-sixth of the altitude, gives the same result as that already found. The application of the rule to the case of cylinders, frustums of cones, spheres, &c., is left as an exercise for the student.

## EXAMPLES.

1. One of the bases of a rectangular prismoid is 25 feet by 20, the other 15 feet by 10, and the altitude 12 feet; required the volume. *Ans.* 3700 cu. ft.

2. What is the volume of a stick of hewn timber, whose ends are 30 inches by 27, and 24 inches by 18, its length being 24 feet? *Ans.* 102 cu. ft.

## MENSURATION OF REGULAR POLYEDRONS.

123. A REGULAR POLYEDRON is a polyedron bounded by equal regular polygons.

The polyedral angles of any regular polyedron are all equal.

124. There are five regular polyedrons (Book VII., Page 208).

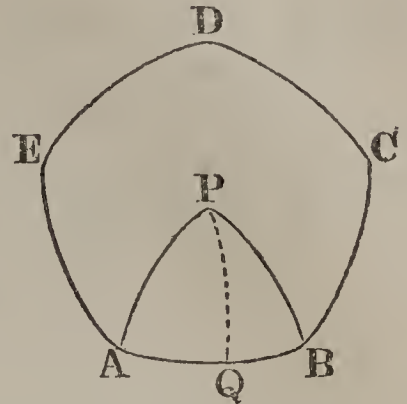
*To find the diedral angle between the faces of a regular polyedron.*

125. Let the vertex of any polyedral angle be taken as the centre of a sphere whose radius is 1: then will this sphere, by its intersections with the faces of the polyedral angle, determine a regular spherical polygon whose sides will be equal to the plane angles that bound the polyedral angle, and whose angles are equal to the diedral angles between the faces.

It only remains to deduce a formula for finding one angle of a regular spherical polygon, when the sides are given.



Let  $ABCDE$  represent a regular spherical polygon, and let  $P$  be the pole of a small circle passing through its vertices. Suppose  $P$  to be connected with each of the vertices by arcs of great circles; there will thus be formed as many equal isosceles triangles as the polygon has sides, the vertical angle in each being equal to  $360^\circ$  divided by the number of sides. Through  $P$  draw  $PQ$  perpendicular to  $AB$ : then will  $AQ$  be equal to  $BQ$ . If we denote the number of sides by  $n$ , the angle  $APQ$  will be equal to  $\frac{360^\circ}{2n}$ , or  $\frac{180^\circ}{n}$ .



In the right-angled spherical triangle  $APQ$ , we know the base  $AQ$ , and the vertical angle  $APQ$ ; hence, by Napier's rules for circular parts, we have,

$$\sin (90^\circ - APQ) = \cos (90^\circ - PAQ) \cos AQ;$$

or, by reduction, denoting the side  $AB$  by  $s$ , and the angle  $PAB$ , by  $A$ ,

$$\cos \frac{180^\circ}{n} = \sin \frac{1}{2}A \cos \frac{1}{2}s;$$

whence,

$$\sin \frac{1}{2}A = \frac{\cos \frac{180^\circ}{n}}{\cos \frac{1}{2}s}.$$

#### EXAMPLES.

In the Tetraedron,

$$\frac{180^\circ}{n} = 60^\circ, \quad \text{and} \quad \frac{1}{2}s = 30^\circ \quad \therefore A = 70^\circ 31' 42''.$$

In the Hexaedron,

$$\frac{180^\circ}{n} = 60^\circ, \quad \text{and} \quad \frac{1}{2}s = 45^\circ \quad \therefore A = 90^\circ.$$

In the Octaedron,

$$\frac{180^\circ}{n} = 45^\circ, \text{ and } \frac{1}{2}s = 30^\circ \therefore A = 109^\circ 28' 18''.$$

In the Dodecaedron,

$$\frac{180^\circ}{n} = 60^\circ, \text{ and } \frac{1}{2}s = 54^\circ \therefore A = 116^\circ 33' 54''.$$

In the Icosaedron,

$$\frac{180^\circ}{n} = 36^\circ, \text{ and } \frac{1}{2}s = 30^\circ \therefore A = 138^\circ 11' 23''.$$

*To find the volume of a regular polyedron.*

126. If planes be passed through the centre of the polyedron and each of the edges, they will divide the polyedron into as many equal right pyramids as the polyedron has faces. The common vertex of these pyramids will be at the centre of the polyedron, their bases will be the faces of the polyedron, and their lateral faces will bisect the diedral angles of the polyedron. The volume of each pyramid will be equal to its base into one-third of its altitude, and this multiplied by the number of faces, will be the volume of the polyedron.

It only remains to deduce a formula for finding the distance from the centre to one face of the polyedron.

Conceive a perpendicular to be drawn from the centre of the polyedron to one face; the foot of this perpendicular will be the centre of the face. From the foot of this perpendicular, draw a perpendicular to either side of the face in which it lies, and connect the point thus determined with the centre of the polyedron. There will thus be formed a right-angled triangle, whose base is the apothem of the face, whose angle at the base is half the diedral angle of the polyedron, and whose altitude is the required altitude of the pyramid, or in other words, the radius of the inscribed sphere.

Denoting the perpendicular by  $P$ , the base by  $b$ , and the diedral angle by  $A$ , we have Formula (3), Art. 37, Trig.,

$$P = b \tan \frac{1}{2}A;$$

but  $b$  is the apothem of one face; if, therefore, we denote the number of sides in that face by  $n$ , and the length of each side by  $s$ , we shall have (Art. 101, Mens.),

$$b = \frac{1}{2}s \cot \frac{180^\circ}{n};$$

whence, by substitution,

$$P = \frac{1}{2}s \cot \frac{180^\circ}{n} \tan \frac{1}{2}A;$$

hence, the volume may be computed. The volumes of all the regular polyedrons have been computed on the supposition that their edges are each equal to 1, and the results are given in the following

TABLE.

NAMES.	NO. OF FACES.	VOLUMES.
Tetraedron, . . . . .	4 . . . . .	0.1178513
Hexaedron, . . . . .	6 . . . . .	1.0000000
Octaedron, . . . . .	8 . . . . .	0.4714045
Dodecaedron, . . . . .	12 . . . . .	7.6631189
Icosaedron, . . . . .	20 . . . . .	2.1816950

From the principles demonstrated in Book VII., we may write the following

## RULE.

*To find the volume of any regular polyedron, multiply the cube of its edge by the corresponding tabular volume; the product will be the volume required.*



## EXAMPLES.

1. What is the volume of a tetraedron, whose edge is 15 ?  
*Ans.* 397.75.
2. What is the volume of a hexaedron, whose edge is 12 ?  
*Ans.* 1728.
3. What is the volume of a octaedron, whose edge is 20 ?  
*Ans.* 3771.236.
4. What is the volume of a dodecadron, whose edge  
is 25 ? *Ans.* 119736.2328.
5. What is the volume of an icosaedron, whose edge  
is 20 ? *Ans.* 17453.56.

# A TABLE

## OF

### LOGARITHMS OF NUMBERS

FROM 1 TO 10,000.

N.	Log.	N.	Log.	N.	Log.	N.	Log.
1	0.000000	26	1.414973	51	1.707570	76	1.880814
2	0.301030	27	1.431354	52	1.716003	77	1.886491
3	0.477121	28	1.447158	53	1.724276	78	1.892095
4	0.602060	29	1.462398	54	1.732394	79	1.897627
5	0.698970	30	1.477121	55	1.740363	80	1.903090
6	0.778151	31	1.491362	56	1.748188	81	1.908485
7	0.845098	32	1.505150	57	1.755875	82	1.913814
8	0.903090	33	1.518514	58	1.763428	83	1.919078
9	0.954243	34	1.531479	59	1.770852	84	1.924279
10	1.000000	35	1.544068	60	1.778151	85	1.929419
11	1.041393	36	1.556303	61	1.785330	86	1.934498
12	1.079181	37	1.568202	62	1.792392	87	1.939519
13	1.113943	38	1.579784	63	1.799341	88	1.944483
14	1.146128	39	1.591065	64	1.806181	89	1.949390
15	1.176091	40	1.602060	65	1.812913	90	1.954243
16	1.204120	41	1.612784	66	1.819544	91	1.959041
17	1.230449	42	1.623249	67	1.826075	92	1.963788
18	1.255273	43	1.633468	68	1.832509	93	1.968483
19	1.278754	44	1.643453	69	1.838849	94	1.973128
20	1.301030	45	1.653213	70	1.845098	95	1.977724
21	1.322219	46	1.662758	71	1.851258	96	1.982271
22	1.342423	47	1.672098	72	1.857333	97	1.986772
23	1.361728	48	1.681241	73	1.863323	98	1.991226
24	1.380211	49	1.690196	74	1.869232	99	1.995635
25	1.397940	50	1.698970	75	1.875061	100	2.000000

REMARK. In the following table, in the nine right hand columns of each page, where the first or leading figures change from 9's to 0's, points or dots are introduced instead of the 0's, to catch the eye, and to indicate that from thence the two figures of the Logarithm to be taken from the second column, stand in the next line below.

N.	0	1	2	3	4	5	6	7	8	9	D.
100	000000	0434	0868	1301	1734	2166	2598	3029	3461	3891	432
101	4321	4751	5181	5609	6038	6466	6894	7321	7748	8174	428
102	8600	9026	9451	9876	*300	*724	1147	1570	1993	2415	424
103	012837	3259	3680	4100	4521	4940	5360	5779	6197	6616	419
104	7033	7451	7868	8284	8700	9116	9532	9947	*361	*775	416
105	021189	1603	2016	2428	2841	3252	3664	4075	4486	4896	412
106	5306	5715	6125	6533	6942	7350	7757	8164	8571	8978	408
107	9384	9789	*195	*600	1004	1408	1812	2216	2619	3021	404
108	033424	3826	4227	4628	5029	5430	5830	6230	6629	7028	400
109	7426	7825	8223	8620	9017	9414	9811	*207	*602	*998	396
110	041393	1787	2182	2576	2969	3362	3755	4148	4540	4932	393
111	5323	5714	6105	6495	6885	7275	7664	8053	8442	8830	389
112	9218	9606	9993	*380	*766	1153	1533	1924	2309	2694	386
113	053078	3463	3846	4230	4613	4996	5378	5760	6142	6524	382
114	6905	7286	7666	8046	8426	8805	9185	9563	9942	*320	379
115	060698	1075	1452	1829	2206	2582	2958	3333	3709	4083	376
116	4458	4832	5206	5580	5953	6326	6699	7071	7443	7815	372
117	8186	8557	8928	9298	9668	*38	*407	*776	1145	1514	369
118	071882	2250	2617	2985	3352	3718	4085	4451	4816	5182	366
119	5547	5912	6276	6640	7004	7368	7731	8094	8457	8819	363
120	079181	9543	9904	*266	*626	*987	1347	1707	2067	2426	360
121	082785	3144	3503	3861	4219	4576	4934	5291	5647	6004	357
122	6360	6716	7071	7426	7781	8136	8490	8845	9198	9552	355
123	9905	*258	*611	*963	1315	1667	2018	2370	2721	3071	351
124	093422	3772	4122	4471	4820	5169	5518	5866	6215	6562	349
125	6910	7257	7604	7951	8298	8644	8990	9335	9681	*26	346
126	100371	0715	1059	1403	1747	2091	2434	2777	3119	3462	343
127	3804	4146	4487	4828	5169	5510	5851	6191	6531	6871	340
128	7210	7549	7888	8227	8565	8903	9241	9579	9916	*253	338
129	110590	0926	1263	1599	1934	2270	2605	2940	3275	3609	335
130	113943	4277	4611	4944	5278	5611	5943	6276	6608	6940	333
131	7271	7603	7934	8265	8595	8926	9256	9586	9915	*245	330
132	120574	0903	1231	1560	1888	2216	2544	2871	3198	3525	328
133	3852	4178	4504	4830	5156	5481	5806	6131	6456	6781	325
134	7105	7429	7753	8076	8399	8722	9045	9368	9690	*12	323
135	130334	0655	0977	1298	1619	1939	2260	2580	2900	3219	321
136	3539	3858	4177	4496	4814	5133	5451	5769	6086	6403	318
137	6721	7037	7354	7671	7987	8303	8618	8934	9249	9564	315
138	9879	*194	*508	*822	1136	1450	1763	2076	2389	2702	314
139	143015	3327	3639	3951	4263	4574	4885	5196	5507	5818	311
140	146128	6438	6748	7058	7367	7676	7985	8294	8603	8911	309
141	9219	9527	9835	*142	*449	*756	1063	1370	1676	1982	307
142	152288	2594	2900	3205	3510	3815	4120	4424	4728	5032	305
143	5336	5640	5943	6246	6549	6852	7154	7457	7759	8061	303
144	8362	8664	8965	9266	9567	9868	*168	*469	*769	1068	301
145	161368	1667	1967	2266	2564	2863	3161	3460	3758	4055	299
146	4353	4650	4947	5244	5541	5838	6134	6430	6726	7022	297
147	7317	7613	7908	8203	8497	8792	9086	9380	9674	9968	295
148	170262	0555	0848	1141	1434	1726	2019	2311	2603	2895	293
149	3186	3478	3769	4060	4351	4641	4932	5222	5512	5802	291
150	176091	6381	6670	6959	7248	7536	7825	8113	8401	8689	289
151	8977	9264	9552	9839	*126	*413	*699	*985	1272	1558	287
152	181844	2129	2415	2700	2985	3270	3555	3839	4123	4407	285
153	4691	4975	5259	5542	5825	6108	6391	6674	6956	7239	283
154	7521	7803	8084	8366	8647	8928	9209	9490	9771	*51	281
155	190332	0612	0892	1171	1451	1730	2010	2289	2567	2846	279
156	3125	3403	3681	3959	4237	4514	4792	5069	5346	5623	278
157	5899	6176	6453	6729	7005	7281	7556	7832	8107	8382	276
158	8657	8932	9206	9481	9755	*29	*303	*577	*850	1124	274
159	201397	1670	1943	2216	2488	2761	3033	3305	3577	3848	272
N.	0	1	2	3	4	5	6	7	8	9	D.



N.	0	1	2	3	4	5	6	7	8	9	D.
160	204120	4391	4663	4934	5204	5475	5746	6016	6286	6556	271
161	6826	7096	7365	7634	7904	8173	8441	8710	8979	9247	269
162	9515	9783	••51	•319	•586	•853	1121	1388	1654	1921	267
163	212188	2454	2720	2986	3252	3518	3783	4049	4314	4579	266
164	4844	5109	5373	5638	5902	6166	6430	6694	6957	7221	264
165	7484	7747	8010	8273	8536	8798	9060	9323	9585	9846	262
166	220108	0370	0631	0892	1153	1414	1675	1936	2196	2456	261
167	2716	2976	3236	3496	3755	4015	4274	4533	4792	5051	259
168	5309	5568	5826	6084	6342	6600	6858	7115	7372	7630	258
169	7887	8144	8400	8657	8913	9170	9426	9682	9938	•193	256
170	230449	0704	0960	1215	1470	1724	1979	2234	2488	2742	254
171	2996	3250	3504	3757	4011	4264	4517	4770	5023	5276	253
172	5528	5781	6033	6285	6537	6789	7041	7292	7544	7795	252
173	8046	8297	8548	8799	9049	9299	9550	9800	••50	•300	250
174	240549	0799	1048	1297	1546	1795	2044	2293	2541	2790	249
175	3038	3286	3534	3782	4030	4277	4525	4772	5019	5266	248
176	5513	5759	6006	6252	6499	6745	6991	7237	7482	7728	246
177	7973	8219	8464	8709	8954	9198	9443	9687	9932	•176	245
178	250420	0664	0908	1151	1395	1638	1881	2125	2368	2610	243
179	2853	3096	3338	3580	3822	4064	4306	4548	4790	5031	242
180	255273	5514	5755	5996	6237	6477	6718	6958	7198	7439	241
181	7679	7918	8158	8398	8637	8877	9116	9355	9594	9833	239
182	260071	0310	0548	0787	1025	1263	1501	1739	1976	2214	238
183	2451	2688	2925	3162	3399	3636	3873	4109	4346	4582	237
184	4818	5054	5290	5525	5761	5996	6232	6467	6702	6937	235
185	7172	7406	7641	7875	8110	8344	8578	8812	9046	9279	234
186	9513	9746	9980	•213	•446	•679	•912	1144	1377	1609	233
187	271842	2074	2306	2538	2770	3001	3233	3464	3696	3927	232
188	4158	4389	4620	4850	5081	5311	5542	5772	6002	6232	230
189	6462	6692	6921	7151	7380	7609	7838	8067	8296	8525	229
190	278754	8982	9211	9439	9667	9895	•123	•351	•578	•806	228
191	281033	1261	1488	1715	1942	2169	2396	2622	2849	3075	227
192	3301	3527	3753	3979	4205	4431	4656	4882	5107	5332	226
193	5557	5782	6007	6232	6456	6681	6905	7130	7354	7578	225
194	7802	8026	8249	8473	8696	8920	9143	9366	9589	9812	223
195	290035	0257	0480	0702	0925	1147	1369	1591	1813	2034	222
196	2256	2478	2699	2920	3141	3363	3584	3804	4025	4246	221
197	4466	4687	4907	5127	5347	5567	5787	6007	6226	6446	220
198	6665	6884	7104	7323	7542	7761	7979	8198	8416	8635	219
199	8853	9071	9289	9507	9725	9943	•161	•378	•595	•813	218
200	301030	1247	1464	1681	1898	2114	2331	2547	2764	2980	217
201	3196	3412	3628	3844	4059	4275	4491	4706	4921	5136	216
202	5351	5566	5781	5996	6211	6425	6639	6854	7068	7282	215
203	7496	7710	7924	8137	8351	8564	8778	8991	9204	9417	213
204	9630	9843	••56	•268	•481	•693	•906	1118	1330	1542	212
205	311754	1966	2177	2389	2600	2812	3023	3234	3445	3656	211
206	3867	4078	4289	4499	4710	4920	5130	5340	5551	5760	210
207	5970	6180	6390	6599	6809	7018	7227	7436	7646	7854	209
208	8063	8272	8481	8689	8898	9106	9314	9522	9730	9938	208
209	320146	0354	0562	0769	0977	1184	1391	1598	1805	2012	207
210	322219	2426	2633	2839	3046	3252	3458	3665	3871	4077	206
211	4282	4488	4694	4899	5105	5310	5516	5721	5926	6131	205
212	6336	6541	6745	6950	7155	7359	7563	7767	7972	8176	204
213	8380	8583	8787	8991	9194	9398	9601	9805	•••8	•211	203
214	330414	0617	0819	1022	1225	1427	1630	1832	2034	2236	202
215	2438	2640	2842	3044	3246	3447	3649	3850	4051	4253	202
216	4454	4655	4856	5057	5257	5458	5658	5859	6059	6260	201
217	6460	6660	6860	7060	7260	7459	7659	7858	8058	8257	200
218	8456	8656	8855	9054	9253	9451	9650	9849	••47	•246	199
219	340444	0642	0841	1039	1237	1435	1632	1830	2028	2225	198
N.	0	1	2	3	4	5	6	7	8	9	D.

N.	0	1	2	3	4	5	6	7	8	9	D.
220	342423	2620	2817	3014	3212	3409	3606	3802	3999	4196	197
221	4392	4589	4785	4981	5178	5374	5570	5766	5962	6157	196
222	6353	6549	6744	6939	7135	7330	7525	7720	7915	8110	195
223	8305	8500	8694	8889	9083	9278	9472	9666	9860	•054	194
224	350248	0442	0636	0829	1023	1216	1410	1603	1796	1989	193
225	2183	2375	2568	2761	2954	3147	3339	3532	3724	3916	193
226	4108	4301	4493	4685	4876	5068	5260	5452	5643	5834	192
227	6026	6217	6408	6599	6790	6981	7172	7363	7554	7744	191
228	7935	8125	8316	8506	8696	8886	9076	9266	9456	9646	190
229	9835	•25	•215	•404	•593	•783	•972	1161	1350	1539	189
230	361728	1917	2105	2294	2482	2671	2859	3048	3236	3424	188
231	3612	3800	3988	4176	4363	4551	4739	4926	5113	5301	188
232	5488	5675	5862	6049	6236	6423	6610	6796	6983	7169	187
233	7356	7542	7729	7915	8101	8287	8473	8659	8845	9030	186
234	9216	9401	9587	9772	9958	•143	•328	•513	•698	•883	185
235	371068	1253	1437	1622	1806	1991	2175	2360	2544	2728	184
236	2912	3096	3280	3464	3647	3831	4015	4198	4382	4565	184
237	4748	4932	5115	5298	5481	5664	5846	6029	6212	6394	183
238	6577	6759	6942	7124	7306	7488	7670	7852	8034	8216	182
239	8398	8580	8761	8943	9124	9306	9487	9668	9849	•030	181
240	380211	0392	0573	0754	0934	1115	1296	1476	1656	1837	181
241	2017	2197	2377	2557	2737	2917	3097	3277	3456	3636	180
242	3815	3995	4174	4353	4533	4712	4891	5070	5249	5428	179
243	5606	5785	5964	6142	6321	6499	6677	6856	7034	7212	178
244	7390	7568	7746	7923	8101	8279	8456	8634	8811	8989	178
245	9166	9343	9520	9698	9875	•051	•228	•405	•582	•759	177
246	390935	1112	1288	1464	1641	1817	1993	2169	2345	2521	176
247	2697	2873	3048	3224	3400	3575	3751	3926	4101	4277	176
248	4452	4627	4802	4977	5152	5326	5501	5676	5850	6025	175
249	6199	6374	6548	6722	6896	7071	7245	7419	7592	7766	174
250	397940	8114	8287	8461	8634	8808	8981	9154	9328	9501	173
251	9674	9847	•020	•192	•365	•538	•711	•883	1056	1228	173
252	401401	1573	1745	1917	2089	2261	2433	2605	2777	2949	172
253	3121	3292	3464	3635	3807	3978	4149	4320	4492	4663	171
254	4834	5005	5176	5346	5517	5688	5858	6029	6199	6370	171
255	6540	6710	6881	7051	7221	7391	7561	7731	7901	8070	170
256	8240	8410	8579	8749	8918	9087	9257	9426	9595	9764	169
257	9933	•102	•271	•440	•609	•777	•946	1114	1283	1451	169
258	411620	1788	1956	2124	2293	2461	2629	2796	2964	3132	168
259	3300	3467	3635	3803	3970	4137	4305	4472	4639	4806	167
260	414973	5140	5307	5474	5641	5808	5974	6141	6308	6474	167
261	6641	6807	6973	7139	7306	7472	7638	7804	7970	8135	166
262	8301	8467	8633	8798	8964	9129	9295	9460	9625	9791	165
263	9956	•121	•286	•451	•616	•781	•945	1110	1275	1439	165
264	421604	1788	1933	2097	2261	2426	2590	2754	2918	3082	164
265	3246	3410	3574	3737	3901	4065	4228	4392	4555	4718	164
266	4882	5045	5208	5371	5534	5697	5860	6023	6186	6349	163
267	6511	6674	6836	6999	7161	7324	7486	7648	7811	7973	162
268	8135	8297	8459	8621	8783	8944	9106	9268	9429	9591	162
269	9752	9914	•075	•236	•398	•559	•720	•881	1042	1203	161
270	431364	1525	1685	1846	2007	2167	2328	2488	2649	2809	161
271	2969	3130	3290	3450	3610	3770	3930	4090	4249	4409	160
272	4569	4729	4888	5048	5207	5367	5526	5685	5844	6004	159
273	6163	6322	6481	6640	6798	6957	7116	7275	7433	7592	159
274	7751	7909	8067	8226	8384	8542	8701	8859	9017	9175	158
275	9333	9491	9648	9806	9964	•122	•279	•437	•594	•752	158
276	440909	1066	1224	1381	1538	1695	1852	2009	2166	2323	157
277	2480	2637	2793	2950	3106	3263	3419	3576	3732	3889	157
278	4045	4201	4357	4513	4669	4825	4981	5137	5293	5449	156
279	5604	5760	5915	6071	6226	6382	6537	6692	6848	7003	155
N.	0	1	2	3	4	5	6	7	8	9	D.



N.	0	1	2	3	4	5	6	7	8	9	D.
280	447158	7313	7468	7623	7778	7933	8088	8242	8397	8552	155
281	8706	8861	9015	9170	9324	9478	9633	9787	9941	••95	154
282	450249	0403	0557	0711	0865	1018	1172	1326	1479	1633	154
283	1785	1940	2093	2247	2400	2553	2706	2859	3012	3165	153
284	3318	3471	3624	3777	3930	4082	4235	4387	4540	4692	153
285	4845	4997	5150	5302	5454	5606	5758	5910	6062	6214	152
286	6366	6518	6670	6821	6973	7125	7276	7428	7579	7731	152
287	7882	8033	8184	8336	8487	8638	8789	8940	9091	9242	151
288	9392	9543	9694	9845	9995	•146	•296	•447	•597	•748	151
289	460898	1048	1198	1348	1499	1649	1799	1948	2098	2248	150
290	462398	2548	2697	2847	2997	3146	3296	3445	3594	3744	150
291	3893	4042	4191	4340	4490	4639	4788	4936	5085	5234	149
292	5383	5532	5680	5829	5977	6126	6274	6423	6571	6719	149
293	6868	7016	7164	7312	7460	7608	7756	7904	8052	8200	148
294	8347	8495	8643	8790	8938	9085	9233	9380	9527	9675	148
295	9822	9969	•116	•263	•410	•557	•704	•851	•998	1145	147
296	471292	1438	1585	1732	1878	2025	2171	2318	2464	2610	146
297	2756	2903	3049	3195	3341	3487	3633	3779	3925	4071	146
298	4216	4362	4508	4653	4799	4944	5090	5235	5381	5526	146
299	5671	5816	5962	6107	6252	6397	6542	6687	6832	6976	145
300	477121	7266	7411	7555	7700	7844	7989	8133	8278	8422	145
301	8566	8711	8855	8999	9143	9287	9431	9575	9719	9863	144
302	480007	0151	0294	0438	0582	0725	0869	1012	1156	1299	144
303	1443	1586	1729	1872	2016	2159	2302	2445	2588	2731	143
304	2874	3016	3159	3302	3445	3587	3730	3872	4015	4157	143
305	4300	4442	4585	4727	4869	5011	5153	5295	5437	5579	142
306	5721	5863	6005	6147	6289	6430	6572	6714	6855	6997	142
307	7138	7280	7421	7563	7704	7845	7986	8127	8269	8410	141
308	8551	8692	8833	8974	9114	9255	9396	9537	9677	9818	141
309	9958	••99	•239	•380	•520	•661	•801	•941	1081	1222	140
310	491362	1502	1642	1782	1922	2062	2201	2341	2481	2621	140
311	2760	2900	3040	3179	3319	3458	3597	3737	3876	4015	139
312	4155	4294	4433	4572	4711	4850	4989	5128	5267	5406	139
313	5544	5683	5822	5960	6099	6238	6376	6515	6653	6791	139
314	6930	7068	7206	7344	7483	7621	7759	7897	8035	8173	138
315	8311	8448	8586	8724	8862	8999	9137	9275	9412	9550	138
316	9687	9824	9962	••99	•236	•374	•511	•648	•785	•922	137
317	501059	1196	1333	1470	1607	1744	1880	2017	2154	2291	137
318	2427	2564	2700	2837	2973	3109	3246	3382	3518	3655	136
319	3791	3927	4063	4199	4335	4471	4607	4743	4878	5014	136
320	505150	5286	5421	5557	5693	5828	5964	6099	6234	6370	136
321	6505	6640	6776	6911	7046	7181	7316	7451	7586	7721	135
322	7856	7991	8126	8260	8395	8530	8664	8799	8934	9068	135
323	9203	9337	9471	9606	9740	9874	•••9	•143	•277	•411	134
324	510545	0679	0813	0947	1081	1215	1349	1482	1616	1750	134
325	1883	2017	2151	2284	2418	2551	2684	2818	2951	3084	133
326	3218	3351	3484	3617	3750	3883	4016	4149	4282	4414	133
327	4548	4681	4813	4946	5079	5211	5344	5476	5609	5741	133
328	5874	6006	6139	6271	6403	6535	6668	6800	6932	7064	132
329	7195	7328	7460	7592	7724	7855	7987	8119	8251	8382	132
330	518514	8646	8777	8909	9040	9171	9303	9434	9566	9697	131
331	9828	9959	••90	•221	•353	•484	•615	•745	•876	1007	131
332	521138	1269	1400	1530	1661	1792	1922	2053	2183	2314	131
333	2444	2575	2705	2835	2966	3096	3226	3356	3486	3616	130
334	3746	3876	4006	4136	4266	4396	4526	4656	4785	4915	130
335	5045	5174	5304	5434	5563	5693	5822	5951	6081	6210	129
336	6339	6469	6598	6727	6856	6985	7114	7243	7372	7501	129
337	7630	7759	7888	8016	8145	8274	8402	8531	8660	8788	129
338	8917	9045	9174	9302	9430	9559	9687	9815	9943	••72	128
339	530200	0328	0456	0584	0712	0840	0968	1096	1223	1351	128
N.	0	1	2	3	4	5	6	7	8	9	D.



## A TABLE OF LOGARITHMS FROM 1 TO 10,000.

N.	0	1	2	3	4	5	6	7	8	9	D.
340	531479	1607	1734	1862	1990	2117	2245	2372	2500	2627	128
341	2754	2882	3009	3136	3264	3391	3518	3645	3772	3899	127
342	4026	4153	4280	4407	4534	4661	4787	4914	5041	5167	127
343	5294	5421	5547	5674	5800	5927	6053	6180	6306	6432	126
344	6558	6685	6811	6937	7063	7189	7315	7441	7567	7693	126
345	7819	7945	8071	8197	8322	8448	8574	8699	8825	8951	126
346	9076	9202	9327	9452	9578	9703	9829	9954	••79	•204	125
347	540329	0455	0580	0705	0830	0955	1080	1205	1330	1454	125
348	1579	1704	1829	1953	2078	2203	2327	2452	2576	2701	125
349	2825	2950	3074	3199	3323	3447	3571	3696	3820	3944	124
350	544068	4192	4316	4440	4564	4688	4812	4936	5060	5183	124
351	5307	5431	5555	5678	5802	5925	6049	6172	6296	6419	124
352	6543	6666	6789	6913	7036	7159	7282	7405	7529	7652	123
353	7775	7898	8021	8144	8267	8389	8512	8635	8758	8881	123
354	9003	9126	9249	9371	9494	9610	9739	9861	9984	•106	123
355	550228	0351	0473	0595	0717	0840	0962	1084	1206	1328	122
356	1450	1572	1694	1816	1938	2060	2181	2303	2425	2547	122
357	2668	2790	2911	3033	3155	3276	3398	3519	3640	3762	121
358	3883	4004	4126	4247	4368	4489	4610	4731	4852	4973	121
359	5094	5215	5336	5457	5578	5699	5820	5940	6061	6182	121
360	556303	6423	6544	6664	6785	6905	7026	7146	7267	7387	120
361	7507	7627	7748	7868	7988	8108	8228	8349	8469	8589	120
362	8709	8829	8948	9068	9188	9308	9428	9548	9667	9787	120
363	9907	•26	•146	•265	•385	•504	•624	•743	•863	•982	119
364	561101	1221	1340	1459	1578	1698	1817	1936	2055	2174	119
365	2293	2412	2531	2650	2769	2887	3006	3125	3244	3362	119
366	3481	3600	3718	3837	3955	4074	4192	4311	4429	4548	119
367	4666	4784	4903	5021	5139	5257	5376	5494	5612	5730	118
368	5843	5966	6084	6202	6320	6437	6555	6673	6791	6909	118
369	7026	7144	7262	7379	7497	7614	7732	7849	7967	8084	118
370	568202	8319	8436	8554	8671	8788	8905	9023	9140	9257	117
371	9374	9491	9608	9725	9842	9959	•76	•193	•309	•426	117
372	570543	0660	0776	0893	1010	1126	1243	1359	1476	1592	117
373	1709	1825	1942	2058	2174	2291	2407	2523	2639	2755	116
374	2872	2988	3104	3220	3336	3452	3568	3684	3800	3915	116
375	4031	4147	4263	4379	4494	4610	4726	4841	4957	5072	116
376	5188	5303	5419	5534	5650	5765	5880	5996	6111	6226	115
377	6341	6457	6572	6687	6802	6917	7032	7147	7262	7377	115
378	7492	7607	7722	7836	7951	8066	8181	8295	8410	8525	115
379	8639	8754	8868	8983	9097	9212	9326	9441	9555	9669	114
380	579784	9898	•12	•126	•241	•355	•469	•583	•697	•811	114
381	580925	1039	1153	1267	1381	1495	1608	1722	1836	1950	114
382	2063	2177	2291	2404	2518	2631	2745	2858	2972	3085	114
383	3199	3312	3426	3539	3652	3765	3879	3992	4105	4218	113
384	4331	4444	4557	4670	4783	4896	5009	5122	5235	5348	113
385	5461	5574	5686	5799	5912	6024	6137	6250	6362	6475	113
386	6587	6700	6812	6925	7037	7149	7262	7374	7486	7599	112
387	7711	7823	7935	8047	8160	8272	8384	8496	8608	8720	112
388	8832	8944	9056	9167	9279	9391	9503	9615	9726	9838	112
389	9950	•61	•173	•284	•396	•507	•619	•730	•842	•953	112
390	591065	1176	1287	1399	1510	1621	1732	1843	1955	2066	111
391	2177	2288	2399	2510	2621	2732	2843	2954	3064	3175	111
392	3286	3397	3508	3618	3729	3840	3950	4061	4171	4282	111
393	4393	4503	4614	4724	4834	4945	5055	5165	5276	5386	110
394	5496	5606	5717	5827	5937	6047	6157	6267	6377	6487	110
395	6597	6707	6817	6927	7037	7146	7256	7366	7476	7586	110
396	7695	7805	7914	8024	8134	8243	8353	8462	8572	8681	110
397	8791	8900	9009	9119	9228	9337	9446	9556	9665	9774	109
398	9883	9992	•101	•210	•319	•428	•537	•646	•755	•864	109
399	600973	1082	1191	1299	1408	1517	1625	1734	1843	1951	109
N.	0	1	2	3	4	5	6	7	8	9	D.

N.	0	1	2	3	4	5	6	7	8	9	D.
400	602060	2169	2277	2386	2494	2603	2711	2819	2928	3036	108
401	3144	3253	3361	3469	3577	3686	3794	3902	4010	4118	108
402	4226	4334	4442	4550	4658	4766	4874	4982	5089	5197	108
403	5305	5413	5521	5628	5736	5844	5951	6059	6166	6274	108
404	6381	6489	6596	6704	6811	6919	7026	7133	7241	7348	107
405	7455	7562	7669	7777	7884	7991	8098	8205	8312	8419	107
406	8526	8633	8740	8847	8954	9061	9167	9274	9381	9488	107
407	9594	9701	9808	9914	••21	•128	•234	•341	•447	•554	107
408	610660	0767	0873	0979	1086	1192	1298	1405	1511	1617	106
409	1723	1829	1936	2042	2148	2254	2360	2466	2572	2678	106
410	612784	2890	2996	3102	3207	3313	3419	3525	3630	3736	106
411	3842	3947	4053	4159	4264	4370	4475	4581	4686	4792	106
412	4897	5003	5108	5213	5319	5424	5529	5634	5740	5845	105
413	5950	6055	6160	6265	6370	6476	6581	6686	6790	6895	105
414	7000	7105	7210	7315	7420	7525	7629	7734	7839	7943	105
415	8048	8153	8257	8362	8466	8571	8676	8780	8884	8989	105
416	9093	9198	9302	9406	9511	9615	9719	9824	9928	••32	104
417	620136	0240	0344	0448	0552	0656	0760	0864	0968	1072	104
418	1176	1280	1384	1488	1592	1695	1799	1903	2007	2110	104
419	2214	2318	2421	2525	2628	2732	2835	2939	3042	3146	104
420	623249	3353	3456	3559	3663	3766	3869	3973	4076	4179	103
421	4282	4385	4488	4591	4695	4798	4901	5004	5107	5210	103
422	5312	5415	5518	5621	5724	5827	5929	6032	6135	6238	103
423	6340	6443	6546	6648	6751	6853	6956	7058	7161	7263	103
424	7366	7468	7571	7673	7775	7878	7980	8082	8185	8287	102
425	8389	8491	8593	8695	8797	8900	9002	9104	9206	9308	102
426	9410	9512	9613	9715	9817	9919	••21	•123	•224	•326	102
427	630428	0530	0631	0733	0835	0936	1038	1139	1241	1342	102
428	1444	1545	1647	1748	1849	1951	2052	2153	2255	2356	101
429	2457	2559	2660	2761	2862	2963	3064	3165	3266	3367	101
430	633468	3569	3670	3771	3872	3973	4074	4175	4276	4376	100
431	4477	4578	4679	4779	4880	4981	5081	5182	5283	5383	100
432	5484	5584	5685	5785	5886	5986	6087	6187	6287	6388	100
433	6488	6588	6688	6789	6889	6989	7089	7189	7290	7390	100
434	7490	7590	7690	7790	7890	7990	8090	8190	8290	8389	99
435	8489	8589	8689	8789	8888	8988	9088	9188	9287	9387	99
436	9486	9586	9686	9785	9885	9984	••84	•183	•283	•382	99
437	640481	0581	0680	0779	0879	0978	1077	1177	1276	1375	99
438	1474	1573	1672	1771	1871	1970	2069	2168	2267	2366	99
439	2465	2563	2662	2761	2860	2959	3058	3156	3255	3354	99
440	643453	3551	3650	3749	3847	3946	4044	4143	4242	4340	98
441	4439	4537	4636	4734	4832	4931	5029	5127	5226	5324	98
442	5422	5521	5619	5717	5815	5913	6011	6110	6208	6306	98
443	6404	6502	6600	6698	6796	6894	6992	7089	7187	7285	98
444	7383	7481	7579	7676	7774	7872	7969	8067	8165	8262	98
445	8360	8458	8555	8653	8750	8848	8945	9043	9140	9237	97
446	9335	9432	9530	9627	9724	9821	9919	••16	•113	•210	97
447	650308	0405	0502	0599	0696	0793	0890	0987	1084	1181	97
448	1278	1375	1472	1569	1666	1762	1859	1956	2053	2150	97
449	2246	2343	2440	2536	2633	2730	2826	2923	3019	3116	97
450	653213	3309	3405	3502	3598	3695	3791	3888	3984	4080	96
451	4177	4273	4369	4465	4562	4658	4754	4850	4946	5042	96
452	5138	5235	5331	5427	5523	5619	5715	5810	5906	6002	96
453	6098	6194	6290	6386	6482	6577	6673	6769	6864	6960	96
454	7056	7152	7247	7343	7438	7534	7629	7725	7820	7916	96
455	8011	8107	8202	8298	8393	8488	8584	8679	8774	8870	95
456	8965	9060	9155	9250	9346	9441	9536	9631	9726	9821	95
457	9916	••11	•106	•201	•296	•391	•486	•581	•676	•771	95
458	660865	0960	1055	1150	1245	1339	1434	1529	1623	1718	95
459	1813	1907	2002	2096	2191	2286	2380	2475	2569	2663	95
N.	0	1	2	3	4	5	6	7	8	9	D.



N.	0	1	2	3	4	5	6	7	8	9	D.
460	662758	2852	2947	3041	3135	3230	3324	3418	3512	3607	94
461	3701	3795	3889	3983	4078	4172	4266	4360	4454	4548	94
462	4642	4736	4830	4924	5018	5112	5206	5299	5393	5487	94
463	5581	5675	5769	5862	5956	6050	6143	6237	6331	6424	94
464	6518	6612	6705	6799	6892	6986	7079	7173	7266	7360	94
465	7453	7546	7640	7733	7826	7920	8013	8106	8199	8293	93
466	8386	8479	8572	8665	8759	8852	8945	9038	9131	9224	93
467	9317	9410	9503	9596	9689	9782	9875	9967	••60	•153	93
468	670246	0339	0431	0524	0617	0710	0802	0895	0988	1080	93
469	1173	1265	1358	1451	1543	1636	1728	1821	1913	2005	93
470	672098	2190	2283	2375	2467	2560	2652	2744	2836	2929	92
471	3021	3113	3205	3297	3390	3482	3574	3666	3758	3850	92
472	3942	4034	4126	4218	4310	4402	4494	4586	4677	4769	92
473	4861	4953	5045	5137	5228	5320	5412	5503	5595	5687	92
474	5778	5870	5962	6053	6145	6236	6328	6419	6511	6602	92
475	6694	6785	6876	6968	7059	7151	7242	7333	7424	7516	91
476	7607	7698	7789	7881	7972	8063	8154	8245	8336	8427	91
477	8518	8609	8700	8791	8882	8973	9064	9155	9246	9337	91
478	9428	9519	9610	9700	9791	9882	9973	••63	•154	•245	91
479	680336	0426	0517	0607	0698	0789	0879	0970	1060	1151	91
480	681241	1332	1422	1513	1603	1693	1784	1874	1964	2055	90
481	2145	2235	2326	2416	2506	2596	2686	2777	2867	2957	90
482	3047	3137	3227	3317	3407	3497	3587	3677	3767	3857	90
483	3947	4037	4127	4217	4307	4396	4486	4576	4666	4756	90
484	4845	4935	5025	5114	5204	5294	5383	5473	5563	5652	90
485	5742	5831	5921	6010	6100	6189	6279	6368	6458	6547	89
486	6636	6726	6815	6904	6994	7083	7172	7261	7351	7440	89
487	7529	7618	7707	7796	7886	7975	8064	8153	8242	8331	89
488	8420	8509	8598	8687	8776	8865	8953	9042	9131	9220	89
489	9309	9398	9486	9575	9664	9753	9841	9930	••19	•107	89
490	690196	0285	0373	0462	0550	0639	0728	0816	0905	0993	89
491	1081	1170	1258	1347	1435	1524	1612	1700	1789	1877	88
492	1965	2053	2142	2230	2318	2406	2494	2583	2671	2759	88
493	2847	2935	3023	3111	3199	3287	3375	3463	3551	3639	88
494	3727	3815	3903	3991	4078	4166	4254	4342	4430	4517	88
495	4605	4693	4781	4868	4956	5044	5131	5219	5307	5394	88
496	5482	5569	5657	5744	5832	5919	6007	6094	6182	6269	87
497	6356	6444	6531	6618	6706	6793	6880	6968	7055	7142	87
498	7229	7317	7404	7491	7578	7665	7752	7839	7926	8014	87
499	8101	8188	8275	8362	8449	8535	8622	8709	8796	8883	87
500	698970	9057	9144	9231	9317	9404	9491	9578	9664	9751	87
501	9838	9924	••11	••98	•184	•271	•358	•444	•531	•617	87
502	700704	0790	0877	0963	1050	1136	1222	1309	1395	1482	86
503	1568	1654	1741	1827	1913	1999	2086	2172	2258	2344	86
504	2431	2517	2603	2689	2775	2861	2947	3033	3119	3205	86
505	3291	3377	3463	3549	3635	3721	3807	3893	3979	4065	86
506	4151	4236	4322	4408	4494	4579	4665	4751	4837	4922	86
507	5008	5094	5179	5265	5350	5436	5522	5607	5693	5778	86
508	5864	5949	6035	6120	6206	6291	6376	6462	6547	6632	85
509	6718	6803	6888	6974	7059	7144	7229	7315	7400	7485	85
510	707570	7655	7740	7826	7911	7996	8081	8166	8251	8336	85
511	8421	8506	8591	8676	8761	8846	8931	9015	9100	9185	85
512	9270	9355	9440	9524	9609	9694	9779	9863	9948	••33	85
513	710117	0202	0287	0371	0456	0540	0625	0710	0794	0879	85
514	0963	1048	1132	1217	1301	1385	1470	1554	1639	1723	84
515	1807	1892	1976	2060	2144	2229	2313	2397	2481	2566	84
516	2650	2734	2818	2902	2986	3070	3154	3238	3323	3407	84
517	3491	3575	3659	3742	3826	3910	3994	4078	4162	4246	84
518	4330	4414	4497	4581	4665	4749	4833	4916	5000	5084	84
519	5167	5251	5335	5418	5502	5586	5669	5753	5836	5920	84
N.	0	1	2	3	4	5	6	7	8	9	D.



## A TABLE OF LOGARITHMS FROM 1 TO 10,000.

9

N.	0	1	2	3	4	5	6	7	8	9	D.
520	716003	6087	6170	6254	6337	6421	6504	6588	6671	6754	83
521	6838	6921	7004	7088	7171	7254	7338	7421	7504	7587	83
522	7671	7754	7837	7920	8003	8086	8169	8253	8336	8419	83
523	8502	8585	8668	8751	8834	8917	9000	9083	9165	9248	83
524	9331	9414	9497	9580	9663	9745	9828	9911	9994	••77	83
525	720159	0242	0325	0407	0490	0573	0655	0738	0821	0903	83
526	0986	1068	1151	1233	1316	1398	1481	1563	1646	1728	82
527	1811	1893	1975	2058	2140	2222	2305	2387	2469	2552	82
528	2634	2716	2798	2881	2963	3045	3127	3209	3291	3374	82
529	3456	3538	3620	3702	3784	3866	3948	4030	4112	4194	82
530	724276	4358	4440	4522	4604	4685	4767	4849	4931	5013	82
531	5095	5176	5258	5340	5422	5503	5585	5667	5748	5830	82
532	5912	5993	6075	6156	6238	6320	6401	6483	6564	6646	82
533	6727	6809	6890	6972	7053	7134	7216	7297	7379	7460	81
534	7541	7623	7704	7785	7866	7948	8029	8110	8191	8273	81
535	8354	8435	8516	8597	8678	8759	8841	8922	9003	9084	81
536	9165	9246	9327	9408	9489	9570	9651	9732	9813	9893	81
537	9974	••55	•136	•217	•298	•378	•459	•540	•621	•702	81
538	730782	0863	0944	1024	1105	1186	1266	1347	1428	1508	81
539	1589	1669	1750	1830	1911	1991	2072	2152	2233	2313	81
540	732394	2474	2555	2635	2715	2796	2876	2956	3037	3117	80
541	3197	3278	3358	3438	3518	3598	3679	3759	3839	3919	80
542	3999	4079	4160	4240	4320	4400	4480	4560	4640	4720	80
543	4800	4880	4960	5040	5120	5200	5279	5359	5439	5519	80
544	5599	5679	5759	5838	5918	5998	6078	6157	6237	6317	80
545	6397	6476	6556	6635	6715	6795	6874	6954	7034	7113	80
546	7193	7272	7352	7431	7511	7590	7670	7749	7829	7908	79
547	7957	8067	8146	8225	8305	8384	8463	8543	8622	8701	79
548	8781	8860	8939	9018	9097	9177	9256	9335	9414	9493	79
549	9572	9651	9731	9810	9889	9968	••47	•126	•205	•284	79
550	740363	0442	0521	0600	0678	0757	0836	0915	0994	1073	79
551	1152	1230	1309	1388	1467	1546	1624	1703	1782	1860	79
552	1939	2018	2096	2175	2254	2332	2411	2489	2568	2647	79
553	2725	2804	2882	2961	3039	3118	3196	3275	3353	3431	78
554	3510	3588	3667	3745	3823	3902	3980	4058	4136	4215	78
555	4293	4371	4449	4528	4606	4684	4762	4840	4919	4997	78
556	5075	5153	5231	5309	5387	5465	5543	5621	5699	5777	78
557	5855	5933	6011	6089	6167	6245	6323	6401	6479	6556	78
558	6634	6712	6790	6868	6945	7023	7101	7179	7256	7334	78
559	7412	7489	7567	7645	7722	7800	7878	7955	8033	8110	78
560	748188	8266	8343	8421	8498	8576	8653	8731	8808	8885	77
561	8963	9040	9118	9195	9272	9350	9427	9504	9582	9659	77
562	9736	9814	9891	9968	••45	•123	•200	•277	•354	•431	77
563	750008	0586	0663	0740	0817	0894	0971	1048	1125	1202	77
564	1279	1356	1433	1510	1587	1664	1741	1818	1895	1972	77
565	2048	2125	2202	2279	2356	2433	2509	2586	2663	2740	77
566	2816	2893	2970	3047	3123	3200	3277	3353	3430	3506	77
567	3583	3660	3736	3813	3889	3966	4042	4119	4195	4272	77
568	4348	4425	4501	4578	4654	4730	4807	4883	4960	5036	76
569	5112	5189	5265	5341	5417	5494	5570	5646	5722	5799	76
570	755875	5951	6027	6103	6180	6256	6332	6408	6484	6560	76
571	6636	6712	6788	6864	6940	7016	7092	7168	7244	7320	76
572	7396	7472	7548	7624	7700	7775	7851	7927	8003	8079	76
573	8155	8230	8306	8382	8458	8533	8609	8685	8761	8836	76
574	8912	8988	9063	9139	9214	9290	9366	9441	9517	9592	76
575	9668	9743	9819	9894	9970	••45	•121	•196	•272	•347	75
576	760422	0498	0573	0649	0724	0799	0875	0950	1025	1101	75
577	1176	1251	1326	1402	1477	1552	1627	1702	1778	1853	75
578	1928	2003	2078	2153	2228	2303	2378	2453	2529	2604	75
579	2579	2754	2829	2904	2978	3053	3128	3203	3278	3353	75
N.	0	1	2	3	4	5	6	7	8	9	D.

N.	0	1	2	3	4	5	6	7	8	9	D.
580	763428	3503	3578	3653	3727	3802	3877	3952	4027	4101	75
581	4176	4251	4326	4400	4475	4550	4624	4699	4774	4848	75
582	4923	4998	5072	5147	5221	5296	5370	5445	5520	5594	75
583	5669	5743	5818	5892	5966	6041	6115	6190	6264	6338	74
584	6413	6487	6562	6636	6710	6785	6859	6933	7007	7082	74
585	7156	7230	7304	7379	7453	7527	7601	7675	7749	7823	74
586	7898	7972	8046	8120	8194	8268	8342	8416	8490	8564	74
587	8638	8712	8786	8860	8934	9008	9082	9156	9230	9303	74
588	9377	9451	9525	9599	9673	9746	9820	9894	9968	•942	74
589	770115	0189	0263	0336	0410	0484	0557	0631	0705	0778	74
590	770852	0926	0999	1073	1146	1220	1293	1367	1440	1514	74
591	1587	1661	1734	1808	1881	1955	2028	2102	2175	2248	73
592	2322	2395	2468	2542	2615	2688	2762	2835	2908	2981	73
593	3055	3128	3201	3274	3348	3421	3494	3567	3640	3713	73
594	3786	3860	3933	4006	4079	4152	4225	4298	4371	4444	73
595	4517	4590	4663	4736	4809	4882	4955	5028	5100	5173	73
596	5246	5319	5392	5465	5538	5610	5683	5756	5829	5902	73
597	5974	6047	6120	6193	6265	6338	6411	6483	6556	6629	73
598	6701	6774	6846	6919	6992	7064	7137	7209	7282	7354	73
599	7427	7499	7572	7644	7717	7789	7862	7934	8006	8079	72
600	778154	8224	8296	8368	8441	8513	8585	8658	8730	8802	72
601	8874	8947	9019	9091	9163	9236	9308	9380	9452	9524	72
602	9596	9669	9741	9813	9885	9957	•929	•101	•173	•245	72
603	780317	0389	0461	0533	0605	0677	0749	0821	0893	0965	72
604	1037	1109	1181	1253	1324	1396	1468	1540	1612	1684	72
605	1755	1827	1899	1971	2042	2114	2186	2258	2329	2401	72
606	2473	2544	2616	2688	2759	2831	2902	2974	3046	3117	72
607	3189	3260	3332	3403	3475	3546	3618	3689	3761	3832	71
608	3904	3975	4046	4118	4189	4261	4332	4403	4475	4546	71
609	4617	4689	4760	4831	4902	4974	5045	5116	5187	5259	71
610	785330	5401	5472	5543	5615	5686	5757	5828	5899	5970	71
611	6041	6112	6183	6254	6325	6396	6467	6538	6609	6680	71
612	6751	6822	6893	6964	7035	7106	7177	7248	7319	7390	71
613	7460	7531	7602	7673	7744	7815	7885	7956	8027	8098	71
614	8163	8239	8310	8381	8451	8522	8593	8663	8734	8804	71
615	8875	8946	9016	9087	9157	9228	9299	9369	9440	9510	71
616	9581	9651	9722	9792	9863	9933	•••4	••74	•144	•215	70
617	790285	0356	0426	0496	0567	0637	0707	0778	0848	0918	70
618	0988	1059	1129	1199	1269	1340	1410	1480	1550	1620	70
619	1691	1761	1831	1901	1971	2041	2111	2181	2252	2322	70
620	792392	2462	2532	2602	2672	2742	2812	2882	2952	3022	70
621	3092	3162	3231	3301	3371	3441	3511	3581	3651	3721	70
622	3790	3860	3930	4000	4070	4139	4209	4279	4349	4418	70
623	4488	4558	4627	4697	4767	4836	4906	4976	5045	5115	70
624	5185	5254	5324	5393	5463	5532	5602	5672	5741	5811	70
625	5880	5949	6019	6088	6158	6227	6297	6366	6436	6505	69
626	6574	6644	6713	6782	6852	6921	6990	7060	7129	7198	69
627	7268	7337	7406	7475	7545	7614	7683	7752	7821	7890	69
628	7960	8029	8098	8167	8236	8305	8374	8443	8513	8582	69
629	8651	8720	8789	8858	8927	8996	9065	9134	9203	9272	69
630	799341	9409	9478	9547	9616	9685	9754	9823	9892	9961	69
631	800029	0098	0167	0236	0305	0373	0442	0511	0580	0648	69
632	0717	0786	0854	0923	0992	1061	1129	1198	1266	1335	69
633	1404	1472	1541	1609	1678	1747	1815	1884	1952	2021	69
634	2089	2158	2226	2295	2363	2432	2500	2568	2637	2705	69
635	2774	2842	2910	2979	3047	3116	3184	3252	3321	3389	68
636	3457	3525	3594	3662	3730	3798	3867	3935	4003	4071	68
637	4139	4208	4276	4344	4412	4480	4548	4616	4685	4753	68
638	4821	4889	4957	5025	5093	5161	5229	5297	5365	5433	68
639	5501	5569	5637	5705	5773	5841	5908	5976	6044	6112	68
N.	0	1	2	3	4	5	6	7	8	9	D.



N.	0	1	2	3	4	5	6	7	8	9	D.
640	806180	6248	6316	6384	6451	6519	6587	6655	6723	6790	68
641	6858	6926	6994	7061	7129	7197	7264	7332	7400	7467	68
642	7535	7603	7670	7738	7806	7873	7941	8008	8076	8143	68
643	8211	8279	8346	8414	8481	8549	8616	8684	8751	8818	67
644	8886	8953	9021	9088	9156	9223	9290	9358	9425	9492	67
645	9560	9627	9694	9762	9829	9896	9964	••31	••98	•165	67
646	810233	0300	0367	0434	0501	0569	0636	0703	0770	0837	67
647	0904	0971	1039	1106	1173	1240	1307	1374	1441	1508	67
648	1575	1642	1709	1776	1843	1910	1977	2044	2111	2178	67
649	2245	2312	2379	2445	2512	2579	2646	2713	2780	2847	67
650	812913	2980	3047	3114	3181	3247	3314	3381	3448	3514	57
651	3581	3648	3714	3781	3848	3914	3981	4048	4114	4181	67
652	4248	4314	4381	4447	4514	4581	4647	4714	4780	4847	67
653	4913	4980	5046	5113	5179	5246	5312	5378	5445	5511	66
654	5578	5644	5711	5777	5843	5910	5976	6042	6109	6175	66
655	6241	6308	6374	6440	6506	6573	6639	6705	6771	6838	66
656	6904	6970	7036	7102	7169	7235	7301	7367	7433	7499	66
657	7565	7631	7698	7764	7830	7896	7962	8028	8094	8160	66
658	8226	8292	8358	8424	8490	8556	8622	8688	8754	8820	66
659	8885	8951	9017	9083	9149	9215	9281	9346	9412	9478	66
660	819544	9610	9676	9741	9807	9873	9939	•••4	••70	•136	66
661	820201	0267	0333	0399	0464	0530	0595	0661	0727	0792	66
662	0858	0924	0989	1055	1120	1186	1251	1317	1382	1448	66
663	1514	1579	1645	1710	1775	1841	1906	1972	2037	2103	65
664	2168	2233	2299	2364	2430	2495	2560	2626	2691	2756	65
665	2822	2887	2952	3018	3083	3148	3213	3279	3344	3409	65
666	3474	3539	3605	3670	3735	3800	3865	3930	3996	4061	65
667	4126	4191	4256	4321	4386	4451	4516	4581	4646	4711	65
668	4776	4841	4906	4971	5036	5101	5166	5231	5296	5361	65
669	5426	5491	5556	5621	5686	5751	5815	5880	5945	6010	65
670	826075	6140	6204	6269	6334	6399	6464	6528	6593	6658	65
671	6723	6787	6852	6917	6981	7046	7111	7175	7240	7305	65
672	7369	7434	7499	7563	7628	7692	7757	7821	7886	7951	65
673	8015	8080	8144	8209	8273	8338	8402	8467	8531	8595	64
674	8660	8724	8789	8853	8918	8982	9046	9111	9175	9239	64
675	9304	9368	9432	9497	9561	9625	9690	9754	9818	9882	64
676	9947	••11	••75	•139	•204	•268	•332	•396	•460	•525	64
677	830589	0653	0717	0781	0845	0909	0973	1037	1102	1166	64
678	1230	1294	1358	1422	1486	1550	1614	1678	1742	1806	64
679	1870	1934	1998	2062	2126	2189	2253	2317	2381	2445	64
680	832509	2573	2637	2700	2764	2828	2892	2956	3020	3083	64
681	3147	3211	3275	3338	3402	3466	3530	3593	3657	3721	64
682	3784	3848	3912	3975	4039	4103	4166	4230	4294	4357	64
683	4421	4484	4548	4611	4675	4739	4802	4866	4929	4993	64
684	5056	5120	5183	5247	5310	5373	5437	5500	5564	5627	63
685	5691	5754	5817	5881	5944	6007	6071	6134	6197	6261	63
686	6324	6387	6451	6514	6577	6641	6704	6767	6830	6894	63
687	6957	7020	7083	7146	7210	7273	7336	7399	7462	7525	63
688	7588	7652	7715	7778	7841	7904	7967	8030	8093	8156	63
689	8219	8282	8345	8408	8471	8534	8597	8660	8723	8786	63
690	838849	8912	8975	9038	9101	9164	9227	9289	9352	9415	63
691	9478	9541	9604	9667	9729	9792	9855	9918	9981	••43	63
692	840106	0169	0232	0294	0357	0420	0482	0545	0608	0671	53
693	0733	0796	0859	0921	0984	1046	1109	1172	1234	1297	63
694	1359	1422	1485	1547	1610	1672	1735	1797	1860	1922	63
695	1985	2047	2110	2172	2235	2297	2360	2422	2484	2547	62
696	2609	2672	2734	2796	2859	2921	2983	3046	3108	3170	62
697	3233	3295	3357	3420	3482	3544	3606	3669	3731	3793	62
698	3855	3918	3980	4042	4104	4166	4229	4291	4353	4415	62
699	4477	4539	4601	4664	4726	4788	4850	4912	4974	5036	62
N.	0	1	2	3	4	5	6	7	8	9	D.



N.	0	1	2	3	4	5	6	7	8	9	D.
700	845098	5160	5222	5284	5346	5408	5470	5532	5594	5656	61
701	5718	5780	5842	5904	5966	6028	6090	6151	6213	6275	62
702	6337	6399	6461	6523	6585	6646	6708	6770	6832	6894	62
703	6955	7017	7079	7141	7202	7264	7326	7388	7449	7511	62
704	7573	7634	7696	7758	7819	7881	7943	8004	8066	8128	62
705	8189	8251	8312	8374	8435	8497	8559	8620	8682	8743	62
706	8805	8866	8928	8989	9051	9112	9174	9235	9297	9358	61
707	9419	9481	9542	9604	9665	9726	9788	9849	9911	9972	61
708	850033	0095	0156	0217	0279	0340	0401	0462	0524	0585	61
709	0646	0707	0769	0830	0891	0952	1014	1075	1136	1197	61
710	851258	1320	1381	1442	1503	1564	1625	1686	1747	1809	61
711	1870	1931	1992	2053	2114	2175	2236	2297	2358	2419	61
712	2480	2541	2602	2663	2724	2785	2846	2907	2968	3029	61
713	3090	3150	3211	3272	3333	3394	3455	3516	3577	3637	61
714	3698	3759	3820	3881	3941	4002	4063	4124	4185	4245	61
715	4306	4367	4428	4488	4549	4610	4670	4731	4792	4852	61
716	4913	4974	5034	5095	5156	5216	5277	5337	5398	5459	61
717	5519	5580	5640	5701	5761	5822	5882	5943	6003	6064	61
718	6124	6185	6245	6306	6366	6427	6487	6548	6608	6668	60
719	6729	6789	6850	6910	6970	7031	7091	7152	7212	7272	60
720	857332	7393	7453	7513	7574	7634	7694	7755	7815	7875	60
721	7935	7995	8056	8116	8176	8236	8297	8357	8417	8477	60
722	8537	8597	8657	8718	8778	8838	8898	8958	9018	9078	60
723	9138	9198	9258	9318	9379	9439	9499	9559	9619	9679	60
724	9739	9799	9859	9918	9978	••38	••98	•158	•218	•278	60
725	860338	0398	0458	0518	0578	0637	0697	0757	0817	0877	60
726	0937	0996	1056	1116	1176	1236	1295	1355	1415	1475	60
727	1534	1594	1654	1714	1773	1833	1893	1952	2012	2072	60
728	2131	2191	2251	2310	2370	2430	2489	2549	2608	2668	60
729	2728	2787	2847	2906	2966	3025	3085	3144	3204	3263	60
730	863323	3382	3442	3501	3561	3620	3680	3739	3799	3858	59
731	3917	3977	4036	4096	4155	4214	4274	4333	4392	4452	59
732	4511	4570	4630	4689	4748	4808	4867	4926	4985	5045	59
733	5104	5163	5222	5282	5341	5400	5459	5519	5578	5637	59
734	5696	5755	5814	5874	5933	5992	6051	6110	6169	6228	59
735	6287	6346	6405	6465	6524	6583	6642	6701	6760	6819	59
736	6878	6937	6996	7055	7114	7173	7232	7291	7350	7409	59
737	7467	7526	7585	7644	7703	7762	7821	7880	7939	7998	59
738	8056	8115	8174	8233	8292	8350	8409	8468	8527	8586	59
739	8644	8703	8762	8821	8879	8938	8997	9056	9114	9173	59
740	869232	9290	9349	9408	9466	9525	9584	9642	9701	9760	59
741	9818	9877	9935	9994	••53	•111	•170	•228	•287	•345	59
742	870404	0462	0521	0579	0638	0696	0755	0813	0872	0930	58
743	0989	1047	1106	1164	1223	1281	1339	1398	1456	1515	58
744	1573	1631	1690	1748	1806	1865	1923	1981	2040	2098	58
745	2156	2215	2273	2331	2389	2448	2506	2564	2622	2681	58
746	2739	2797	2855	2913	2972	3030	3088	3146	3204	3262	58
747	3321	3379	3437	3495	3553	3611	3669	3727	3785	3844	58
748	3902	3960	4018	4076	4134	4192	4250	4308	4366	4424	58
749	4482	4540	4598	4656	4714	4772	4830	4888	4945	5003	58
750	875061	5119	5177	5235	5293	5351	5409	5466	5524	5582	58
751	5640	5698	5756	5813	5871	5929	5987	6045	6102	6160	58
752	6218	6276	6333	6391	6449	6507	6564	6622	6680	6737	58
753	6795	6853	6910	6968	7026	7083	7141	7199	7256	7314	58
754	7371	7429	7487	7544	7602	7659	7717	7774	7832	7889	58
755	7947	8004	8062	8119	8177	8234	8292	8349	8407	8464	57
756	8522	8579	8637	8694	8752	8809	8866	8924	8981	9039	57
757	9096	9153	9211	9268	9325	9383	9440	9497	9555	9612	57
758	9669	9726	9784	9841	9898	9956	••13	••70	•127	•185	57
759	880242	0299	0356	0413	0471	0528	0585	0642	0699	0756	57
N.	0	1	2	3	4	5	6	7	8	9	D.

N.	0	1	2	3	4	5	6	7	8	9	D.
760	880814	0871	0928	0985	1042	1099	1156	1213	1271	1328	57
761	1385	1442	1499	1556	1613	1670	1727	1784	1841	1898	57
762	1955	2012	2069	2126	2183	2240	2297	2354	2411	2468	57
763	2525	2581	2638	2695	2752	2809	2866	2923	2980	3037	57
764	3093	3150	3207	3264	3321	3377	3434	3491	3548	3605	57
765	3661	3718	3775	3832	3888	3945	4002	4059	4115	4172	57
766	4229	4285	4342	4399	4455	4512	4569	4625	4682	4739	57
767	4795	4852	4909	4965	5022	5078	5135	5192	5248	5305	57
768	5361	5418	5474	5531	5587	5644	5700	5757	5813	5870	57
769	5926	5983	6039	6096	6152	6209	6265	6321	6378	6434	56
770	886491	6547	6604	6660	6716	6773	6829	6885	6942	6998	56
771	7054	7111	7167	7223	7280	7336	7392	7449	7505	7561	56
772	7617	7674	7730	7786	7842	7898	7955	8011	8067	8123	56
773	8179	8236	8292	8348	8404	8460	8516	8573	8629	8685	56
774	8741	8797	8853	8909	8965	9021	9077	9134	9190	9246	56
775	9302	9358	9414	9470	9526	9582	9638	9694	9750	9806	56
776	9862	9918	9974	0030	0086	0141	0197	0253	0309	0365	56
777	00421	0477	0533	0589	0645	0700	0756	0812	0868	0924	56
778	0580	1035	1091	1147	1203	1259	1314	1370	1426	1482	56
779	1537	1593	1649	1705	1760	1816	1872	1928	1983	2039	56
780	884295	2150	2206	2262	2317	2373	2429	2484	2540	2595	56
781	2451	2707	2762	2818	2873	2929	2985	3040	3096	3151	56
782	3207	3262	3318	3373	3429	3484	3540	3595	3651	3706	56
783	3762	3817	3873	3928	3984	4039	4094	4150	4205	4261	55
784	4316	4371	4427	4482	4538	4593	4648	4704	4759	4814	55
785	4870	4925	4980	5036	5091	5146	5201	5257	5312	5367	55
786	5423	5478	5533	5588	5644	5699	5754	5809	5864	5920	55
787	5975	6030	6085	6140	6195	6251	6306	6361	6416	6471	55
788	6526	6581	6636	6692	6747	6802	6857	6912	6967	7022	55
789	7077	7132	7187	7242	7297	7352	7407	7462	7517	7572	55
790	797627	7682	7737	7792	7847	7902	7957	8012	8067	8122	55
791	8176	8231	8286	8341	8396	8451	8506	8561	8615	8670	55
792	8725	8780	8835	8890	8944	8999	9054	9109	9164	9218	55
793	9273	9328	9383	9437	9492	9547	9602	9656	9711	9766	55
794	9821	9875	9930	9985	0039	0094	0149	0203	0258	0312	55
795	900367	0422	0476	0531	0586	0640	0695	0749	0804	0859	55
796	0913	0968	1022	1077	1131	1186	1240	1295	1349	1404	55
797	1458	1513	1567	1622	1676	1731	1785	1840	1894	1948	54
798	2003	2057	2112	2166	2221	2275	2329	2384	2438	2492	54
799	2547	2601	2655	2710	2764	2818	2873	2927	2981	3036	54
800	903090	3144	3199	3253	3307	3361	3416	3470	3524	3578	54
801	3633	3687	3741	3795	3849	3904	3958	4012	4066	4120	54
802	4174	4229	4283	4337	4391	4445	4499	4553	4607	4661	54
803	4716	4770	4824	4878	4932	4986	5040	5094	5148	5202	54
804	5256	5310	5364	5418	5472	5526	5580	5634	5688	5742	54
805	5796	5850	5904	5958	6012	6066	6119	6173	6227	6281	54
806	6335	6389	6443	6497	6551	6604	6658	6712	6766	6820	54
807	6874	6927	6981	7035	7089	7143	7196	7250	7304	7358	54
808	7411	7465	7519	7573	7626	7680	7734	7787	7841	7895	54
809	7949	8002	8056	8110	8163	8217	8270	8324	8378	8431	54
810	908485	8539	8592	8646	8699	8753	8807	8860	8914	8967	54
811	9021	9074	9128	9181	9235	9289	9342	9396	9449	9503	54
812	9556	9610	9663	9716	9770	9823	9877	9930	9984	0037	53
813	910091	0144	0197	0251	0304	0358	0411	0464	0518	0571	53
814	0624	0678	0731	0784	0838	0891	0944	0998	1051	1104	53
815	1158	1211	1264	1317	1371	1424	1477	1530	1584	1637	53
816	1690	1743	1797	1850	1903	1956	2009	2063	2116	2169	53
817	2222	2275	2328	2381	2435	2488	2541	2594	2647	2700	53
818	2753	2806	2859	2913	2966	3019	3072	3125	3178	3231	53
819	3284	3337	3390	3443	3496	3549	3602	3655	3708	3761	53
N.	0	1	2	3	4	5	6	7	8	9	D.



N.	0	1	2	3	4	5	6	7	8	9	D.
820	913814	3867	3920	3973	4026	4079	4132	4184	4237	4290	53
821	4343	4306	4449	4502	4555	4608	4660	4713	4766	4819	53
822	4872	4925	4977	5030	5083	5136	5189	5241	5294	5347	53
823	5400	5453	5505	5558	5611	5664	5716	5769	5822	5875	53
824	5927	5980	6033	6085	6138	6191	6243	6296	6349	6401	53
825	6454	6507	6559	6612	6664	6717	6770	6822	6875	6927	53
826	6980	7033	7085	7138	7190	7243	7295	7348	7400	7453	53
827	7506	7558	7611	7663	7716	7768	7820	7873	7925	7978	52
828	8030	8083	8135	8188	8240	8293	8345	8397	8450	8502	52
829	8555	8607	8659	8712	8764	8816	8869	8921	8973	9026	52
830	919078	9130	9183	9235	9287	9340	9392	9444	9496	9549	52
831	9601	9653	9706	9758	9810	9862	9914	9967	••19	••71	52
832	920123	0176	0228	0280	0332	0384	0436	0489	0541	0593	52
833	0645	0697	0749	0801	0853	0906	0958	1010	1062	1114	52
834	1166	1218	1270	1322	1374	1426	1478	1530	1582	1634	52
835	1686	1738	1790	1842	1894	1946	1998	2050	2102	2154	52
836	2206	2258	2310	2362	2414	2466	2518	2570	2622	2674	52
837	2725	2777	2829	2881	2933	2985	3037	3089	3140	3192	52
838	3244	3296	3348	3399	3451	3503	3555	3607	3658	3710	52
839	3762	3814	3865	3917	3969	4021	4072	4124	4176	4228	52
840	924279	4331	4383	4434	4486	4538	4589	4641	4693	4744	52
841	4796	4848	4899	4951	5003	5054	5106	5157	5209	5261	52
842	5312	5364	5415	5467	5518	5570	5621	5673	5725	5776	52
843	5828	5879	5931	5982	6034	6085	6137	6188	6240	6291	51
844	6342	6394	6445	6497	6548	6600	6651	6702	6754	6805	51
845	6857	6908	6959	7011	7062	7114	7165	7216	7268	7319	51
846	7370	7422	7473	7524	7576	7627	7678	7730	7781	7832	51
847	7883	7935	7986	8037	8088	8140	8191	8242	8293	8345	51
848	8396	8447	8498	8549	8601	8652	8703	8754	8805	8857	51
849	8908	8959	9010	9061	9112	9163	9215	9266	9317	9368	51
850	929419	9470	9521	9572	9623	9674	9725	9776	9827	9879	51
851	9930	9981	••32	••83	•134	•185	•236	•287	•338	•389	51
852	930440	0491	0542	0592	0643	0694	0745	0796	0847	0898	51
853	0949	1000	1051	1102	1153	1204	1254	1305	1356	1407	51
854	1458	1509	1560	1610	1661	1712	1763	1814	1865	1915	51
855	1966	2017	2068	2118	2169	2220	2271	2322	2372	2423	51
856	2474	2524	2575	2626	2677	2727	2778	2829	2879	2930	51
857	2981	3031	3082	3133	3183	3234	3285	3335	3386	3437	51
858	3487	3538	3589	3639	3690	3740	3791	3841	3892	3943	51
859	3993	4044	4094	4145	4195	4246	4296	4347	4397	4448	51
860	934498	4549	4599	4650	4700	4751	4801	4852	4902	4953	50
861	5003	5054	5104	5154	5205	5255	5306	5356	5406	5457	50
862	5507	5558	5608	5658	5709	5759	5809	5860	5910	5960	50
863	6011	6061	6111	6162	6212	6262	6313	6363	6413	6463	50
864	6514	6564	6614	6665	6715	6765	6815	6865	6916	6966	50
865	7016	7066	7117	7167	7217	7267	7317	7367	7418	7468	50
866	7518	7568	7618	7668	7718	7769	7819	7869	7919	7969	50
867	8019	8069	8119	8169	8219	8269	8320	8370	8420	8470	50
868	8520	8570	8620	8670	8720	8770	8820	8870	8920	8970	50
869	9020	9070	9120	9170	9220	9270	9320	9369	9419	9469	50
870	939519	9569	9619	9669	9719	9769	9819	9869	9918	9968	50
871	940018	0068	0118	0168	0218	0267	0317	0367	0417	0467	50
872	0516	0566	0616	0666	0716	0765	0815	0865	0915	0964	50
873	1014	1064	1114	1163	1213	1263	1313	1362	1412	1462	50
874	1511	1561	1611	1660	1710	1760	1809	1859	1909	1958	50
875	2008	2058	2107	2157	2207	2256	2306	2355	2405	2455	50
876	2504	2554	2603	2653	2702	2752	2801	2851	2901	2950	50
877	3000	3049	3099	3148	3198	3247	3297	3346	3396	3445	49
878	3495	3544	3593	3643	3692	3742	3791	3841	3890	3939	49
879	3989	4038	4088	4137	4186	4236	4285	4335	4384	4433	49
N.	0	1	2	3	4	5	6	7	8	9	D.



N.	0	1	2	3	4	5	6	7	8	9	D.
880	944483	4532	4581	4631	4680	4729	4779	4828	4877	4927	49
881	4976	5025	5074	5124	5173	5222	5272	5321	5370	5419	49
882	5469	5518	5567	5616	5665	5715	5764	5813	5862	5912	49
883	5961	6010	6059	6108	6157	6207	6256	6305	6354	6403	49
884	6452	6501	6551	6600	6649	6698	6747	6796	6845	6894	49
885	6943	6992	7041	7090	7140	7189	7238	7287	7336	7385	49
886	7434	7483	7532	7581	7630	7679	7728	7777	7826	7875	49
887	7924	7973	8022	8070	8119	8168	8217	8266	8315	8364	49
888	8413	8462	8511	8560	8609	8657	8706	8755	8804	8853	49
889	8902	8951	8999	9048	9097	9146	9195	9244	9292	9341	49
890	949390	9439	9488	9536	9585	9634	9683	9731	9780	9829	49
891	9578	9726	9975	••24	••73	•121	•170	•219	•267	•316	49
892	950365	0414	0462	0511	0560	0608	0657	0706	0754	0803	49
893	0851	0900	0949	0997	1046	1095	1143	1192	1240	1289	49
894	1338	1386	1435	1483	1532	1580	1629	1677	1726	1775	49
895	1823	1872	1920	1969	2017	2066	2114	2163	2211	2260	48
896	2308	2356	2405	2453	2502	2550	2599	2647	2696	2744	48
897	2792	2841	2889	2938	2986	3034	3083	3131	3180	3228	48
898	3276	3325	3373	3421	3470	3518	3566	3615	3663	3711	48
899	3760	3808	3856	3905	3953	4001	4049	4098	4146	4194	48
900	954243	4291	4339	4387	4435	4484	4532	4580	4628	4677	48
901	4725	4773	4821	4869	4918	4966	5014	5062	5110	5158	48
902	5207	5255	5303	5351	5399	5447	5495	5543	5592	5640	48
903	5688	5736	5784	5832	5880	5928	5976	6024	6072	6120	48
904	6168	6216	6265	6313	6361	6409	6457	6505	6553	6601	48
905	6649	6697	6745	6793	6840	6888	6936	6984	7032	7080	48
906	7128	7176	7224	7272	7320	7368	7416	7464	7512	7559	48
907	7607	7655	7703	7751	7799	7847	7894	7942	7990	8038	48
908	8086	8134	8181	8229	8277	8325	8373	8421	8468	8516	48
909	8564	8612	8659	8707	8755	8803	8850	8898	8946	8994	48
910	959041	9089	9137	9185	9232	9280	9328	9375	9423	9471	48
911	9518	9566	9614	9661	9709	9757	9804	9852	9900	9947	48
912	9995	••42	••90	•138	•185	•233	•280	•328	•376	•423	48
913	960471	0518	0566	0613	0661	0709	0756	0804	0851	0899	48
914	0946	0994	1041	1089	1136	1184	1231	1279	1326	1374	47
915	1421	1469	1516	1563	1611	1658	1706	1753	1801	1848	47
916	1895	1943	1990	2038	2085	2132	2180	2227	2275	2322	47
917	2369	2417	2464	2511	2559	2606	2653	2701	2748	2795	47
918	2843	2890	2937	2985	3032	3079	3126	3174	3221	3268	47
919	3316	3363	3410	3457	3504	3552	3599	3646	3693	3741	47
920	963788	3835	3882	3929	3977	4024	4071	4118	4165	4212	47
921	4260	4307	4354	4401	4448	4495	4542	4590	4637	4684	47
922	4731	4778	4825	4872	4919	4966	5013	5061	5108	5155	47
923	5202	5249	5296	5343	5390	5437	5484	5531	5578	5625	47
924	5672	5719	5766	5813	5860	5907	5954	6001	6048	6095	47
925	6142	6189	6236	6283	6329	6376	6423	6470	6517	6564	47
926	6611	6658	6705	6752	6799	6845	6892	6939	6986	7033	47
927	7080	7127	7173	7220	7267	7314	7361	7408	7454	7501	47
928	7548	7595	7642	7688	7735	7782	7829	7875	7922	7969	47
929	8016	8062	8109	8156	8203	8249	8296	8343	8390	8436	47
930	968483	8530	8576	8623	8670	8716	8763	8810	8856	8903	47
931	8959	8995	9043	9090	9136	9183	9229	9276	9323	9369	47
932	9416	9463	9509	9556	9602	9649	9695	9742	9789	9835	47
933	9882	9928	9975	••21	••68	•114	•161	•207	•254	•300	47
934	970347	0393	0440	0486	0533	0579	0626	0672	0719	0765	46
935	0812	0858	0904	0951	0997	1044	1090	1137	1183	1229	46
936	1276	1322	1369	1415	1461	1508	1554	1601	1647	1693	46
937	1749	1786	1832	1879	1925	1971	2018	2064	2110	2157	46
938	2203	2249	2295	2342	2388	2434	2481	2527	2573	2619	46
939	2666	2712	2758	2804	2851	2897	2943	2989	3035	3082	46
N.	0	1	2	3	4	5	6	7	8	9	D.

N.	0	1	2	3	4	5	6	7	8	9	D.
940	973128	3174	3220	3266	3313	3359	3405	3451	3497	3543	46
941	3590	3636	3682	3728	3774	3820	3866	3913	3959	4005	46
942	4051	4097	4143	4189	4235	4281	4327	4374	4420	4466	46
943	4512	4558	4604	4650	4696	4742	4788	4834	4880	4926	46
944	4972	5018	5064	5110	5156	5202	5248	5294	5340	5386	46
945	5432	5478	5524	5570	5616	5662	5707	5753	5799	5845	46
946	5891	5937	5983	6029	6075	6121	6167	6212	6258	6304	46
947	6350	6396	6442	6488	6533	6579	6625	6671	6717	6763	46
948	6808	6854	6900	6946	6992	7037	7083	7129	7175	7220	46
949	7266	7312	7358	7403	7449	7495	7541	7586	7632	7678	46
950	977724	7769	7815	7861	7906	7952	7998	8043	8089	8135	46
951	8181	8226	8272	8317	8363	8409	8454	8500	8546	8591	46
952	8637	8683	8728	8774	8819	8865	8911	8956	9002	9047	46
953	9093	9138	9184	9230	9275	9321	9366	9412	9457	9503	46
954	9548	9594	9639	9685	9730	9776	9821	9867	9912	9958	46
955	980003	0049	0094	0140	0185	0231	0276	0322	0367	0412	45
956	0458	0503	0549	0594	0640	0685	0730	0776	0821	0867	45
957	0912	0957	1003	1048	1093	1139	1184	1229	1275	1320	45
958	1366	1411	1456	1501	1547	1592	1637	1683	1728	1773	45
959	1819	1864	1909	1954	2000	2045	2090	2135	2181	2226	45
960	982271	2316	2362	2407	2452	2497	2543	2588	2633	2678	45
961	2723	2769	2814	2859	2904	2949	2994	3040	3085	3130	45
962	3175	3220	3265	3310	3356	3401	3446	3491	3536	3581	45
963	3626	3671	3716	3762	3807	3852	3897	3942	3987	4032	45
964	4077	4122	4167	4212	4257	4302	4347	4392	4437	4482	45
965	4527	4572	4617	4662	4707	4752	4797	4842	4887	4932	45
966	4977	5022	5067	5112	5157	5202	5247	5292	5337	5382	45
967	5426	5471	5516	5561	5606	5651	5696	5741	5786	5830	45
968	5875	5920	5965	6010	6055	6100	6144	6189	6234	6279	45
969	6324	6369	6413	6458	6503	6548	6593	6637	6682	6727	45
970	986772	6817	6861	6906	6951	6996	7040	7085	7130	7175	45
971	7219	7264	7309	7353	7398	7443	7488	7532	7577	7622	45
972	7666	7711	7756	7800	7845	7890	7934	7979	8024	8068	45
973	8113	8157	8202	8247	8291	8336	8381	8425	8470	8514	45
974	8559	8604	8648	8693	8737	8782	8826	8871	8916	8960	45
975	9005	9049	9094	9138	9183	9227	9272	9316	9361	9405	45
976	9450	9494	9539	9583	9628	9672	9717	9761	9806	9850	44
977	9895	9939	9983	••28	••72	•117	•161	•206	•250	•294	44
978	990339	0383	0428	0472	0516	0561	0605	0650	0694	0738	44
979	•0783	0827	0871	0916	0960	1004	1049	1093	1137	1182	44
980	991226	1270	1315	1359	1403	1448	1492	1536	1580	1625	44
981	1669	1713	1758	1802	1846	1890	1935	1979	2023	2067	44
982	2111	2156	2200	2244	2288	2333	2377	2421	2465	2509	44
983	2554	2598	2642	2686	2730	2774	2819	2863	2907	2951	44
984	2995	3039	3083	3127	3172	3216	3260	3304	3348	3392	44
985	3436	3480	3524	3568	3613	3657	3701	3745	3789	3833	44
986	3877	3921	3965	4009	4053	4097	4141	4185	4229	4273	44
987	4317	4361	4405	4449	4493	4537	4581	4625	4669	4713	44
988	4757	4801	4845	4889	4933	4977	5021	5065	5108	5152	44
989	5196	5240	5284	5328	5372	5416	5460	5504	5547	5591	44
990	995635	5679	5723	5767	5811	5854	5898	5942	5986	6030	44
991	6074	6117	6161	6205	6249	6293	6337	6380	6424	6468	44
992	6512	6555	6599	6643	6687	6731	6774	6818	6862	6906	44
993	6949	6993	7037	7080	7124	7168	7212	7255	7299	7343	44
994	7386	7430	7474	7517	7561	7605	7648	7692	7736	7779	44
995	7823	7867	7910	7954	7998	8041	8085	8129	8172	8216	44
996	8259	8303	8347	8390	8434	8477	8521	8564	8608	8652	44
997	8695	8739	8782	8826	8869	8913	8956	9000	9043	9087	44
998	9131	9174	9218	9261	9305	9348	9392	9435	9479	9522	44
999	9565	9609	9652	9696	9739	9783	9826	9870	9913	9957	43
N.	0	1	2	3	4	5	6	7	8	9	D.



A TABLE  
OF  
LOGARITHMIC  
SINES AND TANGENTS  
FOR EVERY  
DEGREE AND MINUTE  
OF THE QUADRANT.

---

REMARK. The minutes in the left-hand column of each page, increasing downwards, belong to the degrees at the top; and those increasing upwards, in the right-hand column, belong to the degrees below.



M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	0.000000		10.000000		0.000000		Infinite.	60
1	6.463726	5017.17	000000	.00	6.463726	5017.17	13.536274	59
2	764756	2934.85	000000	.00	764756	2934.83	235244	58
3	940847	2082.31	000000	.00	940847	2082.31	059153	57
4	7.065786	1615.17	000000	.00	7.065786	1615.17	12.934214	56
5	162696	1319.68	000000	.00	162696	1319.69	837304	55
6	241877	1115.75	9.999999	.01	241878	1115.78	758122	54
7	308824	966.53	999999	.01	308825	996.53	691175	53
8	366816	852.54	999999	.01	366817	852.54	633183	52
9	417968	762.63	999999	.01	417970	762.63	582030	51
10	463725	689.88	999998	.01	463727	689.88	536273	50
11	7.505118	629.81	9.999998	.01	7.505120	629.81	12.494880	49
12	542906	579.36	999997	.01	542909	579.33	457091	48
13	577668	536.41	999997	.01	577672	536.42	422328	47
14	609853	499.38	999996	.01	609857	499.39	390143	46
15	639816	467.14	999996	.01	639820	467.15	360180	45
16	667845	438.81	999995	.01	667849	438.82	332151	44
17	694173	413.72	999995	.01	694179	413.73	305821	43
18	718997	391.35	999994	.01	719004	391.36	280997	42
19	742477	371.27	999993	.01	742484	371.28	257516	41
20	764754	353.15	999993	.01	764761	351.36	235239	40
21	7.785943	336.72	9.999992	.01	7.785951	336.73	12.214049	39
22	806146	321.75	999991	.01	806155	321.76	193845	38
23	825451	308.05	999990	.01	825460	308.06	174540	37
24	843934	295.47	999989	.02	843944	295.49	156056	36
25	861662	283.88	999988	.02	861674	283.90	138326	35
26	878695	273.17	999988	.02	878708	273.18	121292	34
27	895085	263.23	999987	.02	895099	263.25	104901	33
28	910879	253.99	999986	.02	910894	254.01	089106	32
29	926119	245.38	999985	.02	926134	245.40	073866	31
30	940842	237.33	999983	.02	940858	237.35	059142	30
31	7.955082	229.80	9.999982	.02	7.955100	229.81	12.044900	29
32	968870	222.73	999981	.02	968889	222.75	031111	28
33	982233	216.08	999980	.02	982253	216.10	017747	27
34	995198	209.81	999979	.02	995219	209.83	004781	26
35	8.007787	203.90	999977	.02	8.007809	203.92	11.992191	25
36	020021	198.31	999976	.02	020045	198.33	979955	24
37	031919	193.02	999975	.02	031945	193.05	968055	23
38	043501	188.01	999973	.02	043527	188.03	956473	22
39	054781	183.25	999972	.02	054809	183.27	945191	21
40	065776	178.72	999971	.02	065806	178.74	934194	20
41	8.076500	174.41	9.999969	.02	8.076531	174.44	11.923469	19
42	086965	170.31	999968	.02	086997	170.34	913003	18
43	097183	166.39	999966	.02	097217	166.42	902783	17
44	107167	162.65	999964	.03	107202	162.68	892797	16
45	116926	159.08	999963	.03	116963	159.10	883037	15
46	126471	155.66	999961	.03	126510	155.68	873490	14
47	135810	152.38	999959	.03	135851	152.41	864149	13
48	144953	149.24	999958	.03	144996	149.27	855004	12
49	153907	146.22	999955	.03	153952	146.27	846048	11
50	162681	143.33	999954	.03	162727	143.36	837273	10
51	8.171280	140.54	9.999952	.03	8.171328	140.57	11.828672	9
52	179713	137.86	999950	.03	179763	137.90	820237	8
53	187985	135.29	999948	.03	188036	135.32	811064	7
54	196102	132.80	999946	.03	196156	132.84	803844	6
55	204070	130.41	999944	.03	204126	130.44	795874	5
56	211895	128.10	999942	.04	211953	128.14	788047	4
57	219581	125.87	999940	.04	219641	125.90	780359	3
58	227134	123.72	999938	.04	227195	123.76	772805	2
59	234557	121.64	999936	.04	234621	121.68	765379	1
60	241855	119.63	999934	.04	241921	119.67	758079	0
	Cosine	D.	Sine		Cotang.	D.	Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	8.241855	119.63	9.999934	.04	8.241921	119.67	11.58079	60
1	249033	117.68	999932	.04	249102	117.72	750898	59
2	256094	115.80	999929	.04	256165	115.84	743835	58
3	263042	113.98	999927	.04	263115	114.02	736885	57
4	269881	112.21	999925	.04	269956	112.25	730044	56
5	276614	110.50	999922	.04	276691	110.54	723309	55
6	283243	108.83	999920	.04	283323	108.87	716677	54
7	289773	107.21	999918	.04	289856	107.26	710144	53
8	296207	105.65	999915	.04	296292	105.70	703708	52
9	302546	104.13	999913	.04	302634	104.18	697366	51
10	308794	102.66	999910	.04	308884	102.70	691116	50
11	8.314904	101.22	9.999907	.04	8.315046	101.26	11.684954	49
12	321027	99.82	999905	.04	321122	99.87	678878	48
13	327016	98.47	999902	.04	327114	98.51	672886	47
14	332924	97.14	999899	.05	333025	97.19	666975	46
15	338753	95.86	999897	.05	338856	95.90	661144	45
16	344504	94.60	999894	.05	344610	94.65	655390	44
17	350181	93.38	999891	.05	350289	93.43	649711	43
18	355783	92.19	999888	.05	355895	92.24	644105	42
19	361315	91.03	999885	.05	361430	91.08	638570	41
20	366777	89.90	999882	.05	366895	89.95	633105	40
21	8.372171	88.80	9.999879	.05	8.372292	88.85	11.627708	39
22	377499	87.72	999876	.05	377622	87.77	622378	38
23	382762	86.67	999873	.05	382889	86.72	617111	37
24	387962	85.64	999870	.05	388092	85.70	611908	36
25	393101	84.64	999867	.05	393234	84.70	606766	35
26	398179	83.66	999864	.05	398315	83.71	601685	34
27	403199	82.71	999861	.05	403338	82.76	596662	33
28	408161	81.77	999858	.05	408304	81.82	591696	32
29	413068	80.86	999854	.05	413213	80.91	586787	31
30	417919	79.96	999851	.06	418068	80.02	581932	30
31	8.422717	79.09	9.999848	.06	8.422869	79.14	11.577131	29
32	427462	78.23	999844	.06	427618	78.30	572382	28
33	432156	77.40	999841	.06	432315	77.45	567685	27
34	436800	76.57	999838	.06	436962	76.63	563038	26
35	441394	75.77	999834	.06	441560	75.83	558440	25
36	445941	74.99	999831	.06	446110	75.05	553890	24
37	450440	74.22	999827	.06	450613	74.28	549387	23
38	454893	73.46	999823	.06	455070	73.52	544930	22
39	459301	72.73	999820	.06	459481	72.79	540519	21
40	463665	72.00	999816	.06	463849	72.06	536151	20
41	8.467985	71.29	9.999812	.06	8.468172	71.35	11.531828	19
42	472263	70.60	999809	.06	472454	70.66	527546	18
43	476498	69.91	999805	.06	476693	69.98	523307	17
44	480693	69.24	999801	.06	480892	69.31	519108	16
45	484848	68.59	999797	.07	485050	68.65	514950	15
46	488963	67.94	999793	.07	489170	68.01	510830	14
47	493040	67.31	999790	.07	493250	67.38	506750	13
48	497078	66.69	999786	.07	497293	66.76	502707	12
49	501080	66.08	999782	.07	501298	66.15	498702	11
50	505045	65.48	999778	.07	505267	65.55	494733	10
51	8.508974	64.89	9.999774	.07	8.509260	64.96	11.490800	9
52	512867	64.31	999769	.07	513098	64.39	486902	8
53	516726	63.75	999765	.07	516961	63.82	483039	7
54	520551	63.19	999761	.07	520790	63.26	479210	6
55	524343	62.64	999757	.07	524586	62.72	475414	5
56	528102	62.11	999753	.07	528349	62.18	471651	4
57	531828	61.58	999748	.07	532080	61.65	467920	3
58	535523	61.06	999744	.07	535779	61.13	464221	2
59	539186	60.55	999740	.07	539447	60.62	460553	1
60	542819	60.04	999735	.07	543084	60.12	456916	0
	Cosine	D.	Sine		Cotang.	D.	Tang	



M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	8.542819	60.04	9.999735	.07	8.543084	60.12	11.456916	60
1	546422	59.55	999731	.07	546691	59.62	453309	59
2	549995	59.06	999726	.07	550268	59.14	449732	58
3	553539	58.58	999722	.08	553817	58.66	446183	57
4	557054	58.11	999717	.08	557336	58.19	442664	56
5	560540	57.65	999713	.08	560828	57.73	439172	55
6	563999	57.19	999708	.08	564291	57.27	435709	54
7	567431	56.74	999704	.08	567727	56.82	432273	53
8	570836	56.30	999699	.08	571137	56.38	428863	52
9	574214	55.87	999694	.08	574520	55.95	425480	51
10	577566	55.44	999689	.08	577877	55.52	422123	50
11	8.580892	55.02	9.999685	.08	8.581208	55.10	11.418792	49
12	584193	54.60	999680	.08	584514	54.68	415486	48
13	587469	54.19	999675	.08	587795	54.27	412205	47
14	590721	53.79	999670	.08	591051	53.87	408949	46
15	593948	53.39	999665	.08	594283	53.47	405717	45
16	597152	53.00	999660	.08	597492	53.08	402508	44
17	600332	52.61	999655	.08	600677	52.70	399323	43
18	603489	52.23	999650	.08	603839	52.32	396161	42
19	606623	51.86	999645	.09	606978	51.94	393022	41
20	609734	51.49	999640	.09	610094	51.58	389906	40
21	8.612823	51.12	9.999635	.09	8.613189	51.21	11.386811	39
22	615891	50.76	999629	.09	616262	50.85	383738	38
23	618937	50.41	999624	.09	619313	50.50	380687	37
24	621962	50.06	999619	.09	622343	50.15	377657	36
25	624965	49.72	999614	.09	625352	49.81	374648	35
26	627948	49.38	999608	.09	628340	49.47	371660	34
27	630911	49.04	999603	.09	631308	49.13	368692	33
28	633854	48.71	999597	.09	634256	48.80	365744	32
29	636776	48.39	999592	.09	637184	48.48	362816	31
30	639680	48.06	999586	.09	640093	48.16	359907	30
31	8.642563	47.75	9.999581	.09	8.642982	47.84	11.357018	29
32	645428	47.43	999575	.09	645853	47.53	354147	28
33	648274	47.12	999570	.09	648704	47.22	351296	27
34	651102	46.82	999564	.09	651537	46.91	348463	26
35	653911	46.52	999558	.10	654352	46.61	345648	25
36	656702	46.22	999553	.10	657149	46.31	342851	24
37	659475	45.92	999547	.10	659928	46.02	340072	23
38	662230	45.63	999541	.10	662689	45.73	337311	22
39	664968	45.35	999535	.10	665433	45.44	334567	21
40	667689	45.06	999529	.10	668160	45.26	331840	20
41	8.670393	44.79	9.999524	.10	8.670870	44.88	11.329130	19
42	673080	44.51	999518	.10	673563	44.61	326437	18
43	675751	44.24	999512	.10	676239	44.34	323761	17
44	678405	43.97	999506	.10	678900	44.17	321100	16
45	681043	43.70	999500	.10	681544	43.80	318456	15
46	683665	43.44	999493	.10	684172	43.54	315828	14
47	686272	43.18	999487	.10	686784	43.28	313216	13
48	688863	42.92	999481	.10	689381	43.03	310619	12
49	691438	42.67	999475	.10	691963	42.77	308037	11
50	693998	42.42	999469	.10	694529	42.52	305471	10
51	8.696543	42.17	9.999463	.11	8.697081	42.28	11.302919	9
52	699073	41.92	999456	.11	699617	42.03	300383	8
53	701589	41.68	999450	.11	702139	41.79	297861	7
54	704090	41.44	999443	.11	704646	41.55	295354	6
55	706577	41.21	999437	.11	707140	41.32	292860	5
56	709049	40.97	999431	.11	709618	41.08	290382	4
57	711507	40.74	999424	.11	712083	40.85	287917	3
58	713952	40.51	999418	.11	714534	40.62	285465	2
59	716383	40.29	999411	.11	716972	40.40	283028	1
60	718800	40.06	999404	.11	719396	40.17	280604	0
	Cosine	D.	Sine		Cotang.	D.	Tang.	M.



M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	8.718800	40.06	9.999404	.11	8.719396	40.17	11.280604	60
1	721204	39.84	999398	.11	721806	39.95	278194	59
2	723595	39.62	999391	.11	724204	39.74	275796	58
3	725972	39.41	999384	.11	726588	39.52	273412	57
4	728337	39.19	999378	.11	728959	39.30	271041	56
5	730688	38.98	999371	.11	731317	39.09	268683	55
6	733027	38.77	999364	.12	733663	38.89	266337	54
7	735354	38.57	999357	.12	735996	38.68	264004	53
8	737667	38.36	999350	.12	738317	38.48	261683	52
9	739969	38.16	999343	.12	740626	38.27	259374	51
10	742259	37.96	999336	.12	742922	38.07	257078	50
11	8.744536	37.76	9.999329	.12	8.745207	37.87	11.254793	49
12	746802	37.56	999322	.12	747479	37.68	252521	48
13	749055	37.37	999315	.12	749740	37.49	250260	47
14	751297	37.17	999308	.12	751989	37.29	248011	46
15	753528	36.98	999301	.12	754227	37.10	245773	45
16	755747	36.79	999294	.12	756453	36.92	243547	44
17	757955	36.61	999286	.12	758668	36.73	241332	43
18	760151	36.42	999279	.12	760872	36.55	239128	42
19	762337	36.24	999272	.12	763065	36.36	236935	41
20	764511	36.06	999265	.12	765246	36.18	234754	40
21	8.766675	35.88	9.999257	.12	8.767417	36.00	11.232583	39
22	768828	35.70	999250	.13	769578	35.83	230422	38
23	770970	35.53	999242	.13	771727	35.65	228273	37
24	773101	35.35	999235	.13	773866	35.48	226134	36
25	775223	35.18	999227	.13	775995	35.31	224005	35
26	777333	35.01	999220	.13	778114	35.14	221886	34
27	779434	34.84	999212	.13	780222	34.97	219778	33
28	781524	34.67	999205	.13	782320	34.80	217680	32
29	783605	34.51	999197	.13	784408	34.64	215592	31
30	785675	34.31	999189	.13	786486	34.47	213514	30
31	8.787736	34.18	9.999181	.13	8.788554	34.31	11.211446	29
32	789787	34.02	999174	.13	790613	34.15	209387	28
33	791828	33.86	999166	.13	792662	33.99	207338	27
34	793859	33.70	999158	.13	794701	33.83	205299	26
35	795881	33.54	999150	.13	796731	33.68	203269	25
36	797894	33.39	999142	.13	798752	33.52	201248	24
37	799897	33.23	999134	.13	800763	33.37	199237	23
38	801892	33.08	999126	.13	802765	33.22	197235	22
39	803876	32.93	999118	.13	804758	33.07	195242	21
40	805852	32.78	999110	.13	806742	32.92	193258	20
41	8.807819	32.63	9.999102	.13	8.808717	32.78	11.191283	19
42	809777	32.49	999094	.14	810683	32.62	189317	18
43	811726	32.34	999086	.14	812641	32.48	187359	17
44	813667	32.19	999077	.14	814589	32.33	185411	16
45	815599	32.05	999069	.14	816529	32.19	183471	15
46	817522	31.91	999061	.14	818461	32.05	181539	14
47	819436	31.77	999053	.14	820384	31.91	179616	13
48	821343	31.63	999044	.14	822298	31.77	177702	12
49	823240	31.49	999036	.14	824205	31.63	175795	11
50	825130	31.35	999027	.14	826103	31.50	173897	10
51	8.827011	31.22	9.999019	.14	8.827992	31.36	11.172008	9
52	828884	31.08	999010	.14	829874	31.23	170126	8
53	830749	30.95	999002	.14	831748	31.10	168252	7
54	832607	30.82	998993	.14	833613	30.96	166387	6
55	834456	30.69	998984	.14	835471	30.83	164529	5
56	836297	30.56	998976	.14	837321	30.70	162679	4
57	838130	30.43	998967	.15	839163	30.57	160837	3
58	839956	30.30	998958	.15	840998	30.45	159002	2
59	841774	30.17	998950	.15	842825	30.32	157175	1
60	843585	30.00	998941	.15	844644	30.19	155356	0
	Cosine	D.	Sine		Cotang.	D.	Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	8.843585	30.05	9.998941	.15	8.844644	30.19	11.155356	60
1	845387	29.92	998932	.15	846455	30.07	153545	59
2	847183	29.80	998923	.15	848260	29.95	151740	58
3	848971	29.67	998914	.15	850057	29.82	149943	57
4	850751	29.55	998905	.15	851846	29.70	148154	56
5	852525	29.43	998896	.15	853628	29.58	146372	55
6	854291	29.31	998887	.15	855403	29.46	144597	54
7	856049	29.19	998878	.15	857171	29.35	142829	53
8	857801	29.07	998869	.15	858932	29.23	141068	52
9	859546	28.96	998860	.15	860686	29.11	139314	51
10	861283	28.84	998851	.15	862433	29.00	137567	50
11	8.863014	28.73	9.998841	.15	8.864173	28.88	11.135827	49
12	864738	28.61	998832	.15	865906	28.77	134094	48
13	866455	28.50	998823	.16	867632	28.66	132368	47
14	868165	28.39	998813	.16	869351	28.54	130649	46
15	869868	28.28	998804	.16	871064	28.43	128936	45
16	871565	28.17	998795	.16	872770	28.32	127230	44
17	873255	28.06	998785	.16	874469	28.21	125531	43
18	874938	27.95	998776	.16	876162	28.11	123838	42
19	876615	27.86	998766	.16	877849	28.00	122151	41
20	878285	27.73	998757	.16	879529	27.89	120471	40
21	8.879949	27.63	9.998747	.16	8.881202	27.79	11.118798	39
22	881607	27.52	998738	.16	882869	27.68	117131	38
23	883258	27.42	998728	.16	884530	27.58	115470	37
24	884903	27.31	998718	.16	886185	27.47	113815	36
25	886542	27.21	998708	.16	887833	27.37	112167	35
26	888174	27.11	998699	.16	889476	27.27	110524	34
27	889801	27.00	998689	.16	891112	27.17	108888	33
28	891421	26.90	998679	.16	892742	27.07	107258	32
29	893035	26.80	998669	.17	894366	26.97	105634	31
30	894643	26.70	998659	.17	895984	26.87	104016	30
31	8.896246	26.60	9.998649	.17	8.897596	26.77	11.102404	29
32	897842	26.51	998639	.17	899203	26.67	100797	28
33	899432	26.41	998629	.17	900803	26.58	099197	27
34	901017	26.31	998619	.17	902398	26.48	097602	26
35	902596	26.22	998609	.17	903987	26.38	096013	25
36	904169	26.12	998599	.17	905570	26.29	094430	24
37	905736	26.03	998589	.17	907147	26.20	092853	23
38	907297	25.93	998578	.17	908719	26.10	091281	22
39	908853	25.84	998568	.17	910285	26.01	089715	21
40	910404	25.75	998558	.17	911846	25.92	088154	20
41	8.911949	25.66	9.998548	.17	8.913401	25.83	11.086599	19
42	913488	25.56	998537	.17	914951	25.74	085049	18
43	915022	25.47	998527	.17	916495	25.65	083505	17
44	916550	25.38	998516	.18	918034	25.56	081966	16
45	918073	25.29	998506	.18	919568	25.47	080432	15
46	919591	25.20	998495	.18	921096	25.38	078904	14
47	921103	25.12	998485	.18	922619	25.30	077381	13
48	922610	25.03	998474	.18	924136	25.21	075864	12
49	924112	24.94	998464	.18	925649	25.12	074351	11
50	925609	24.86	998453	.18	927156	25.03	072844	10
51	8.927100	24.77	9.998442	.18	8.928658	24.95	11.071342	9
52	928587	24.69	998431	.18	930155	24.86	069845	8
53	930068	24.60	998421	.18	931647	24.78	068353	7
54	931544	24.52	998410	.18	933134	24.70	066866	6
55	933015	24.43	998399	.18	934616	24.61	065384	5
56	934481	24.35	998388	.18	936093	24.53	063907	4
57	935942	24.27	998377	.18	937565	24.45	062435	3
58	937398	24.19	998366	.18	939032	24.37	060968	2
59	938850	24.11	998355	.18	940494	24.30	059506	1
60	940296	24.03	998344	.18	941952	24.21	058048	0
	Cosine	D.	Sine		Cotang.	D.	Tang.	M.



M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	8.940296	24.03	9.998344	19	8.941952	24.21	11.058048	60
1	94.138	23.94	998333	19	943404	24.13	056596	59
2	943174	23.87	998322	19	944852	24.05	055148	58
3	944606	23.79	998311	19	946295	23.97	053705	57
4	946034	23.71	998300	19	947734	23.90	052266	56
5	947456	23.63	998289	19	949168	23.82	050832	55
6	948874	23.55	998277	19	950597	23.74	049403	54
7	950287	23.48	998266	19	952021	23.66	047979	53
8	951696	23.40	998255	19	953441	23.60	046559	52
9	953100	23.32	998243	19	954856	23.51	045144	51
10	954499	23.25	998232	19	956267	23.44	043733	50
11	8.955894	23.17	9.998220	19	8.957674	23.37	11.042326	49
12	957284	23.10	998209	19	959075	23.29	040925	48
13	958670	23.02	998197	19	960473	23.23	039527	47
14	960052	22.95	998186	19	961866	23.14	038134	46
15	961429	22.88	998174	19	963255	23.07	036745	45
16	962801	22.80	998163	19	964639	23.00	035361	44
17	964170	22.73	998151	19	966019	22.93	033981	43
18	965534	22.66	998139	20	967394	22.86	032606	42
19	966893	22.59	998128	20	968766	22.79	031234	41
20	968249	22.52	998116	20	970133	22.71	029867	40
21	8.969600	22.44	9.998104	20	8.971496	22.65	11.028504	39
22	970947	22.38	998092	20	972855	22.57	027145	38
23	972289	22.31	998080	20	974209	22.51	025791	37
24	973628	22.24	998068	20	975560	22.44	024440	36
25	974962	22.17	998056	20	976906	22.37	023094	35
26	976293	22.10	998044	20	978248	22.30	021752	34
27	977619	22.03	998032	20	979586	22.23	020414	33
28	978941	21.97	998020	20	980921	22.17	019079	32
29	980259	21.90	998008	20	982251	22.10	017749	31
30	981573	21.83	997996	20	983577	22.04	016423	30
31	8.982883	21.77	9.997985	20	8.984899	21.97	11.015101	29
32	984189	21.70	997972	20	986217	21.91	013783	28
33	985491	21.63	997959	20	987532	21.84	012468	27
34	986789	21.57	997947	20	988842	21.78	011158	26
35	988083	21.50	997935	21	990149	21.71	009851	25
36	989374	21.44	997922	21	991451	21.65	008549	24
37	990660	21.38	997910	21	992750	21.58	007250	23
38	991943	21.31	997897	21	994045	21.52	005955	22
39	993222	21.25	997885	21	995337	21.46	004663	21
40	994497	21.19	997872	21	996624	21.40	003376	20
41	8.995768	21.12	9.997860	21	8.997908	21.34	11.002092	19
42	997036	21.06	997847	21	999188	21.27	000812	18
43	998299	21.00	997835	21	9.000465	21.21	10.999535	17
44	999560	20.94	997822	21	001738	21.15	998262	16
45	9.000816	20.87	997809	21	003007	21.09	996993	15
46	002069	20.82	997797	21	004272	21.03	995728	14
47	003318	20.76	997784	21	005534	20.97	994466	13
48	004563	20.70	997771	21	006792	20.91	993208	12
49	005805	20.64	997758	21	008047	20.85	991953	11
50	007044	20.58	997745	21	009298	20.80	990702	10
51	9.008278	20.52	9.997732	21	9.010546	20.74	10.989454	9
52	009510	20.46	997719	21	011790	20.68	988210	8
53	010737	20.40	997706	21	013031	20.62	986969	7
54	011962	20.34	997693	22	014268	20.56	985732	6
55	013182	20.29	997680	22	015502	20.51	984498	5
56	014400	20.23	997667	22	016732	20.45	983268	4
57	015613	20.17	997654	22	017959	20.40	982041	3
58	016824	20.12	997641	22	019183	20.33	980817	2
59	018031	20.06	997628	22	020403	20.28	979597	1
60	019235	20.00	997614	22	021620	20.23	978380	0
	Cosine	D.	Sine		Cotang.	D.	Tang.	M.



M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9.019235	20.00	9.997614	.22	9.021620	20.23	10.978380	60
1	020435	19.95	997601	.22	022834	20.17	977166	59
2	021632	19.89	997588	.22	024044	20.11	975956	58
3	022825	19.84	997574	.22	025251	20.06	974749	57
4	024016	19.78	997561	.22	026455	20.00	973545	56
5	025203	19.73	997547	.22	027655	19.95	972345	55
6	026386	19.67	997534	.23	028852	19.90	971148	54
7	027567	19.62	997520	.23	030046	19.85	969954	53
8	028744	19.57	997507	.23	031237	19.79	968763	52
9	029918	19.51	997493	.23	032425	19.74	967575	51
10	031089	19.47	997480	.23	033609	19.69	966391	50
11	9.032257	19.41	9.997466	.23	9.034791	19.64	10.965209	49
12	033421	19.36	997452	.23	035969	19.58	964031	48
13	034582	19.30	997439	.23	037144	19.53	962856	47
14	035741	19.25	997425	.23	038316	19.48	961684	46
15	036896	19.20	997411	.23	039485	19.43	960515	45
16	038048	19.15	997397	.23	040651	19.38	959349	44
17	039197	19.10	997383	.23	041813	19.33	958187	43
18	040342	19.05	997369	.23	042973	19.28	957027	42
19	041485	18.99	997355	.23	044130	19.23	955870	41
20	042625	18.94	997341	.23	045284	19.18	954716	40
21	9.043762	18.89	9.997327	.24	9.046434	19.13	10.953566	39
22	044895	18.84	997313	.24	047582	19.08	952418	38
23	046026	18.79	697299	.24	048727	19.03	951273	37
24	047154	18.75	997285	.24	049869	18.98	950131	36
25	048279	18.70	997271	.24	051008	18.93	948992	35
26	049400	18.65	997257	.24	052144	18.89	947856	34
27	050519	18.60	997242	.24	053277	18.84	946723	33
28	051635	18.55	997228	.24	054407	18.79	945593	32
29	052749	18.50	997214	.24	055535	18.74	944465	31
30	053859	18.45	997199	.24	056659	18.70	943341	30
31	9.054966	18.41	9.997185	.24	9.057781	18.65	10.942219	29
32	056071	18.36	997170	.24	058900	18.60	941100	28
33	057172	18.31	997156	.24	060016	18.55	939984	27
34	058271	18.27	997141	.24	061130	18.51	938870	26
35	059367	18.22	997127	.24	062240	18.46	937760	25
36	060460	18.17	997112	.24	063348	18.42	936652	24
37	061551	18.13	997098	.24	064453	18.37	935547	23
38	062639	18.08	997083	.25	065556	18.33	934444	22
39	063724	18.04	997068	.25	066655	18.28	933345	21
40	064806	17.99	997053	.25	067752	18.24	932248	20
41	9.065885	17.94	9.997039	.25	9.068846	18.19	10.931154	19
42	066962	17.90	997024	.25	069938	18.15	930062	18
43	068036	17.86	997009	.25	071027	18.10	928973	17
44	069107	17.81	996994	.25	072113	18.06	927887	16
45	070176	17.77	996979	.25	073197	18.02	926803	15
46	071242	17.72	996964	.25	074278	17.97	925722	14
47	072306	17.68	996949	.25	075356	17.93	924644	13
48	073366	17.63	996934	.25	076432	17.89	923568	12
49	074424	17.59	996919	.25	077505	17.84	922495	11
50	075480	17.55	996904	.25	078576	17.80	921424	10
51	9.076533	17.50	9.996889	.25	9.079644	17.76	10.920356	9
52	077583	17.46	996874	.25	080710	17.72	919290	8
53	078631	17.42	996858	.25	081773	17.67	918227	7
54	079676	17.38	996843	.25	082833	17.63	917167	6
55	080719	17.33	996828	.25	083891	17.59	916109	5
56	081759	17.29	996812	.26	084947	17.55	915053	4
57	082797	17.25	996797	.26	086000	17.51	914000	3
58	083832	17.21	996782	.26	087050	17.47	912950	2
59	084864	17.17	996766	.26	088098	17.43	911902	1
60	085894	17.13	996751	.26	089144	17.38	910856	0
	Cosine	D.	Sine		Cotang.	D.	Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9.085894	17.13	9.996751	.26	9.089144	17.38	10.910856	60
1	086922	17.09	996735	.26	090187	17.34	909813	59
2	087947	17.04	996720	.26	091228	17.30	908772	58
3	088970	17.00	996704	.26	092266	17.27	907734	57
4	089990	16.96	996688	.26	093302	17.22	906698	56
5	091008	16.92	996673	.26	094336	17.19	905664	55
6	092024	16.88	996657	.26	095367	17.15	904633	54
7	093037	16.84	996641	.26	096395	17.11	903605	53
8	094047	16.80	996625	.26	097422	17.07	902578	52
9	095056	16.76	996610	.26	098446	17.03	901554	51
10	096062	16.73	996594	.26	099468	16.99	900532	50
11	9.097065	16.68	9.996578	.27	9.100487	16.95	10.899513	49
12	098066	16.65	996562	.27	101504	16.91	898496	48
13	099065	16.61	996546	.27	102519	16.87	897481	47
14	100062	16.57	996530	.27	103532	16.84	896468	46
15	101056	16.53	996514	.27	104542	16.80	895458	45
16	102048	16.49	996498	.27	105550	16.76	894450	44
17	103037	16.45	996482	.27	106556	16.72	893444	43
18	104025	16.41	996465	.27	107559	16.69	892441	42
19	105010	16.38	996449	.27	108560	16.65	891440	41
20	105992	16.34	996433	.27	109559	16.61	890441	40
21	9.106973	16.30	9.996417	.27	9.110556	16.58	10.889444	39
22	107951	16.27	996400	.27	111551	16.54	888449	38
23	108927	16.23	996384	.27	112543	16.50	887457	37
24	109901	16.19	996368	.27	113533	16.46	886467	36
25	110873	16.16	996351	.27	114521	16.43	885479	35
26	111842	16.12	996335	.27	115507	16.39	884493	34
27	112809	16.08	996318	.27	116491	16.36	883509	33
28	113774	16.05	996302	.28	117472	16.32	882528	32
29	114737	16.01	996285	.28	118452	16.29	881548	31
30	115698	15.97	996269	.28	119429	16.25	880571	30
31	9.116656	15.94	9.996252	.28	9.120404	16.22	10.879596	29
32	117613	15.90	996235	.28	121377	16.18	878623	28
33	118567	15.87	996219	.28	122348	16.15	877652	27
34	119519	15.83	996202	.28	123317	16.11	876683	26
35	120469	15.80	996185	.28	124284	16.07	875716	25
36	121417	15.76	996168	.28	125249	16.04	874751	24
37	122362	15.73	996151	.28	126211	16.01	873789	23
38	123306	15.69	996134	.28	127172	15.97	872828	22
39	124248	15.66	996117	.28	128130	15.94	871870	21
40	125187	15.62	996100	.28	129087	15.91	870913	20
41	9.126125	15.59	9.996083	.29	9.130041	15.87	10.869959	19
42	127060	15.56	996066	.29	130994	15.84	869006	18
43	127993	15.52	996049	.29	131944	15.81	868056	17
44	128925	15.49	996032	.29	132893	15.77	867107	16
45	129854	15.45	996015	.29	133839	15.74	866161	15
46	130781	15.42	995998	.29	134784	15.71	865216	14
47	131706	15.39	995980	.29	135726	15.67	864274	13
48	132630	15.35	995963	.29	136667	15.64	863333	12
49	133551	15.32	995946	.29	137605	15.61	862395	11
50	134470	15.29	995928	.29	138542	15.58	861458	10
51	9.135387	15.25	9.995911	.29	9.139476	15.55	10.860524	9
52	136303	15.22	995894	.29	140409	15.51	859591	8
53	137216	15.19	995876	.29	141340	15.48	858660	7
54	138128	15.16	995859	.29	142269	15.45	857731	6
55	139037	15.12	995841	.29	143196	15.42	856804	5
56	139944	15.09	995823	.29	144121	15.39	855879	4
57	140850	15.06	995806	.29	145044	15.35	854956	3
58	141754	15.03	995788	.29	145966	15.32	854034	2
59	142655	15.00	995771	.29	146885	15.29	853115	1
60	143555	14.96	995753	.29	147803	15.26	852197	0
	Cosine	D.	Sine		Cotang.	D.	Tang.	M.



M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9.143555	14.96	9.995753	.30	9.147803	15.26	10.852197	60
1	144453	14.93	995735	.30	148718	15.23	851282	59
2	145349	14.90	995717	.30	149632	15.20	850368	58
3	146243	14.87	995699	.30	150544	15.17	849456	57
4	147136	14.84	995681	.30	151454	15.14	848546	56
5	148026	14.81	995664	.30	152363	15.11	847637	55
6	148915	14.78	995646	.30	153269	15.08	846731	54
7	149802	14.75	995628	.30	154174	15.05	845826	53
8	150686	14.72	995610	.30	155077	15.02	844923	52
9	151569	14.69	995591	.30	155978	14.99	844022	51
10	152451	14.66	995573	.30	156877	14.96	843123	50
11	9.153330	14.63	9.995555	.30	9.157775	14.93	10.842225	49
12	154208	14.60	995537	.30	158671	14.90	841329	48
13	155083	14.57	995519	.30	159565	14.87	840435	47
14	155957	14.54	995501	.31	160457	14.84	839543	46
15	156830	14.51	995482	.31	161347	14.81	838653	45
16	157700	14.48	995464	.31	162236	14.79	837764	44
17	158569	14.45	995446	.31	163123	14.76	836877	43
18	159435	14.42	995427	.31	164008	14.73	835992	42
19	160301	14.39	995409	.31	164892	14.70	835108	41
20	161164	14.36	995390	.31	165774	14.67	834226	40
21	9.162025	14.33	9.995372	.31	9.166654	14.64	10.833346	39
22	162885	14.30	995353	.31	167532	14.61	832468	38
23	163743	14.27	995334	.31	168409	14.58	831591	37
24	164600	14.24	995316	.31	169284	14.55	830716	36
25	165454	14.22	995297	.31	170157	14.53	829843	35
26	166307	14.19	995278	.31	171029	14.50	828971	34
27	167159	14.16	995260	.31	171899	14.47	828101	33
28	168008	14.13	995241	.32	172767	14.44	827233	32
29	168856	14.10	995222	.32	173634	14.42	826366	31
30	169702	14.07	995203	.32	174499	14.39	825501	30
31	9.170547	14.05	9.995184	.32	9.175362	14.36	10.824638	29
32	171389	14.02	995165	.32	176224	14.33	823776	28
33	172230	13.99	995146	.32	177084	14.31	822916	27
34	173070	13.96	995127	.32	177942	14.28	822058	26
35	173908	13.94	995108	.32	178799	14.25	821201	25
36	174744	13.91	995089	.32	179655	14.23	820345	24
37	175578	13.88	995070	.32	180508	14.20	819492	23
38	176411	13.86	995051	.32	181360	14.17	818640	22
39	177242	13.83	995032	.32	182211	14.15	817789	21
40	178072	13.80	995013	.32	183059	14.12	816941	20
41	9.178900	13.77	9.994993	.32	9.183907	14.09	10.816093	19
42	179726	13.74	994974	.32	184752	14.07	815248	18
43	180551	13.72	994955	.32	185597	14.04	814403	17
44	181374	13.69	994935	.32	186439	14.02	813561	16
45	182196	13.66	994916	.33	187280	13.99	812720	15
46	183016	13.64	994896	.33	188120	13.96	811880	14
47	183834	13.61	994877	.33	188958	13.93	811042	13
48	184651	13.59	994857	.33	189794	13.91	810206	12
49	185466	13.56	994838	.33	190629	13.89	809371	11
50	186280	13.53	994818	.33	191462	13.86	808538	10
51	9.187092	13.51	9.994798	.33	9.192294	13.84	10.807706	9
52	187903	13.48	994779	.33	193124	13.81	806876	8
53	188712	13.46	994759	.33	193953	13.79	806047	7
54	189519	13.43	994739	.33	194780	13.76	805220	6
55	190325	13.41	994719	.33	195606	13.74	804394	5
56	191130	13.38	994700	.33	196430	13.71	803570	4
57	191933	13.36	994680	.33	197253	13.69	802747	3
58	192734	13.33	994660	.33	198074	13.66	801926	2
59	193534	13.30	994640	.33	198894	13.64	801106	1
60	194332	13.28	994620	.33	199713	13.61	800287	0
	Cosine	D.	Sine		Cotang.	D.	Tang.	M.



M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9.194332	13.28	9.994620	.33	9.199713	13.61	10.800287	6c
1	195129	13.26	994600	.33	200529	13.59	799471	59
2	195925	13.23	994580	.33	201345	13.56	798655	58
3	196719	13.21	994560	.34	202159	13.54	797841	57
4	197511	13.18	994540	.34	202971	13.52	797029	56
5	198302	13.16	994519	.34	203782	13.49	796218	55
6	199091	13.13	994499	.34	204592	13.47	795408	54
7	199879	13.11	994479	.34	205400	13.45	794600	53
8	200666	13.08	994459	.34	206207	13.42	793793	52
9	201451	13.06	994438	.34	207013	13.40	792987	51
10	202234	13.04	994418	.34	207817	13.38	792183	50
11	9.203017	13.01	9.994397	.34	9.208619	13.35	10.791381	49
12	203797	12.99	994377	.34	209420	13.33	790580	48
13	204577	12.96	994357	.34	210220	13.31	789780	47
14	205354	12.94	994336	.34	211018	13.28	788982	46
15	206131	12.92	994316	.34	211815	13.26	788185	45
16	206906	12.89	994295	.34	212611	13.24	787389	44
17	207679	12.87	994274	.35	213405	13.21	786595	43
18	208452	12.85	994254	.35	214198	13.19	785802	42
19	209222	12.82	994233	.35	214989	13.17	785011	41
20	209992	12.80	994212	.35	215780	13.15	784220	40
21	9.210760	12.78	9.994191	.35	9.216568	13.12	10.783432	39
22	211526	12.75	994171	.35	217356	13.10	782644	38
23	212291	12.73	994150	.35	218142	13.08	781858	37
24	213055	12.71	994129	.35	218926	13.05	781074	36
25	213818	12.68	994108	.35	219710	13.03	780290	35
26	214579	12.66	994087	.35	220492	13.01	779508	34
27	215338	12.64	994066	.35	221272	12.99	778728	33
28	216097	12.61	994045	.35	222052	12.97	777948	32
29	216854	12.59	994024	.35	222830	12.94	777170	31
30	217609	12.57	994003	.35	223606	12.92	776394	30
31	9.218363	12.55	9.993981	.35	9.224382	12.90	10.775618	29
32	219116	12.53	993960	.35	225156	12.88	774844	28
33	219868	12.50	993939	.35	225929	12.86	774071	27
34	220618	12.48	993918	.35	226700	12.84	773300	26
35	221367	12.46	993896	.36	227471	12.81	772529	25
36	222115	12.44	993875	.36	228239	12.79	771761	24
37	222861	12.42	993854	.36	229007	12.77	770993	23
38	223606	12.39	993832	.36	229773	12.75	770227	22
39	224349	12.37	993811	.36	230539	12.73	769461	21
40	225092	12.35	993789	.36	231302	12.71	768698	20
41	9.225833	12.33	9.993768	.36	9.232065	12.69	10.767935	19
42	226573	12.31	993746	.36	232826	12.67	767174	18
43	227311	12.28	993725	.36	233586	12.65	766414	17
44	228048	12.26	993703	.36	234345	12.62	765655	16
45	228784	12.24	993681	.36	235103	12.60	764897	15
46	229518	12.22	993660	.36	235859	12.58	764141	14
47	230252	12.20	993638	.36	236614	12.56	763386	13
48	230984	12.18	993616	.36	237368	12.54	762632	12
49	231714	12.16	993594	.37	238120	12.52	761880	11
50	232444	12.14	993572	.37	238872	12.50	761128	10
51	9.233172	12.12	9.993550	.37	9.239622	12.48	10.760378	9
52	233899	12.09	993528	.37	240371	12.46	759629	8
53	234625	12.07	993506	.37	241118	12.44	758882	7
54	235349	12.05	993484	.37	241865	12.42	758135	6
55	236073	12.03	993462	.37	242610	12.40	757390	5
56	236795	12.01	993440	.37	243354	12.38	756646	4
57	237515	11.99	993418	.37	244097	12.36	755903	3
58	238235	11.97	993396	.37	244839	12.34	755161	2
59	238953	11.95	993374	.37	245579	12.32	754421	1
60	239670	11.93	993351	.37	246319	12.30	753681	0
	Cosine	D.	Sine		Cotang.	D.	Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9.239670	11.93	9.993351	.37	9.246319	12.30	10.753681	60
1	240386	11.91	993329	.37	247057	12.28	752943	59
2	241101	11.89	993307	.37	247794	12.26	752206	58
3	241814	11.87	993285	.37	248530	12.24	751470	57
4	242526	11.85	993262	.37	249264	12.22	750736	56
5	243237	11.83	993240	.37	249998	12.20	750002	55
6	243947	11.81	993217	.38	250730	12.18	749270	54
7	244656	11.79	993195	.38	251461	12.17	748539	53
8	245363	11.77	993172	.38	252191	12.15	747809	52
9	246069	11.75	993149	.38	252920	12.13	747080	51
10	246775	11.73	993127	.38	253648	12.11	746352	50
11	9.247478	11.71	9.993104	.38	9.254374	12.09	10.745626	49
12	248181	11.69	993081	.38	255100	12.07	744900	48
13	248883	11.67	993059	.38	255824	12.05	744176	47
14	249583	11.65	993036	.38	256547	12.03	743453	46
15	250282	11.63	993013	.38	257269	12.01	742731	45
16	250980	11.61	992990	.38	257990	12.00	742010	44
17	251677	11.59	992967	.38	258710	11.98	741290	43
18	252373	11.58	992944	.38	259429	11.96	740571	42
19	253067	11.56	992921	.38	260146	11.94	739854	41
20	253761	11.54	992898	.38	260863	11.92	739137	40
21	9.254453	11.52	9.992875	.38	9.261578	11.90	10.738422	39
22	255144	11.50	992852	.38	262292	11.89	737708	38
23	255834	11.48	992829	.39	263005	11.87	736995	37
24	256523	11.46	992806	.39	263717	11.85	736283	36
25	257211	11.44	992783	.39	264428	11.83	735572	35
26	257898	11.42	992759	.39	265138	11.81	734862	34
27	258583	11.41	992736	.39	265847	11.79	734153	33
28	259268	11.39	992713	.39	266555	11.78	733445	32
29	259951	11.37	992690	.39	267261	11.76	732739	31
30	260633	11.35	992666	.39	267967	11.74	732033	30
31	9.261314	11.33	9.992643	.39	9.268671	11.72	10.731329	29
32	261994	11.31	992619	.39	269375	11.70	730625	28
33	262673	11.30	992596	.39	270077	11.69	729923	27
34	263351	11.28	992572	.39	270779	11.67	729221	26
35	264027	11.26	992549	.39	271479	11.65	728521	25
36	264703	11.24	992525	.39	272178	11.64	727822	24
37	265377	11.22	992501	.39	272876	11.62	727124	23
38	266051	11.20	992478	.40	273573	11.60	726427	22
39	266723	11.19	992454	.40	274269	11.58	725731	21
40	267395	11.17	992430	.40	274964	11.57	725036	20
41	9.268065	11.15	9.992406	.40	9.275658	11.55	10.724342	19
42	268734	11.13	992382	.40	276351	11.53	723649	18
43	269402	11.11	992359	.40	277043	11.51	722957	17
44	270069	11.10	992335	.40	277734	11.50	722266	16
45	270735	11.08	992311	.40	278424	11.48	721576	15
46	271400	11.06	992287	.40	279113	11.47	720887	14
47	272064	11.05	992263	.40	279801	11.45	720199	13
48	272726	11.03	992239	.40	280488	11.43	719512	12
49	273388	11.01	992214	.40	281174	11.41	718826	11
50	274049	10.99	992190	.40	281858	11.40	718142	10
51	9.274708	10.98	9.992166	.40	9.282542	11.38	10.717458	9
52	275367	10.96	992142	.40	283225	11.36	716775	8
53	276024	10.94	992117	.41	283907	11.35	716093	7
54	276681	10.92	992093	.41	284588	11.33	715412	6
55	277337	10.91	992069	.41	285268	11.31	714732	5
56	277991	10.89	992044	.41	285947	11.30	714053	4
57	278644	10.87	992020	.41	286624	11.28	713376	3
58	279297	10.86	991996	.41	287301	11.26	712699	2
59	279948	10.84	991971	.41	287977	11.25	712023	1
60	280599	10.82	991947	.41	288652	11.23	711348	0
	Cosine	D.	Sine		Cotang.	D.	Tang.	M.



M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9.280599	10.82	9.991947	.41	9.288652	11.23	10.711348	60
1	281248	10.81	991922	.41	289326	11.22	710674	59
2	281897	10.79	991897	.41	289999	11.20	710001	58
3	282544	10.77	991873	.41	290671	11.18	709329	57
4	283190	10.76	991848	.41	291342	11.17	708658	56
5	283836	10.74	991823	.41	292013	11.15	707987	55
6	284480	10.72	991799	.41	292682	11.14	707318	54
7	285124	10.71	991774	.42	293350	11.12	706650	53
8	285766	10.69	991749	.42	294017	11.11	705983	52
9	286408	10.67	991724	.42	294684	11.09	705316	51
10	287048	10.66	991699	.42	295349	11.07	704651	50
11	9.287687	10.64	9.991674	.42	9.296013	11.06	10.703987	49
12	288326	10.63	991649	.42	296677	11.04	703323	48
13	288964	10.61	991624	.42	297339	11.03	702661	47
14	289600	10.59	991599	.42	298001	11.01	701999	46
15	290236	10.58	991574	.42	298662	11.00	701338	45
16	290870	10.56	991549	.42	299322	10.98	700678	44
17	291504	10.54	991524	.42	299980	10.96	700020	43
18	292137	10.53	991498	.42	300638	10.95	699362	42
19	292768	10.51	991473	.42	301295	10.93	698705	41
20	293399	10.50	991448	.42	301951	10.92	698049	40
21	9.294029	10.48	9.991422	.42	9.302607	10.90	10.697393	39
22	294658	10.46	991397	.42	303261	10.89	696739	38
23	295286	10.45	991372	.43	303914	10.87	696086	37
24	295913	10.43	991346	.43	304567	10.86	695433	36
25	296539	10.42	991321	.43	305218	10.84	694782	35
26	297164	10.40	991295	.43	305869	10.83	694131	34
27	297788	10.39	991270	.43	306519	10.81	693481	33
28	298412	10.37	991244	.43	307168	10.80	692832	32
29	299034	10.36	991218	.43	307815	10.78	692185	31
30	299655	10.34	991193	.43	308463	10.77	691537	30
31	9.300276	10.32	9.991167	.43	9.309109	10.75	10.690891	29
32	300895	10.31	991141	.43	309754	10.74	690246	28
33	301514	10.29	991115	.43	310398	10.73	689602	27
34	302132	10.28	991090	.43	311042	10.71	688958	26
35	302748	10.26	991064	.43	311685	10.70	688315	25
36	303364	10.25	991038	.43	312327	10.68	687673	24
37	303979	10.23	991012	.43	312967	10.67	687033	23
38	304593	10.22	990986	.43	313608	10.65	686392	22
39	305207	10.20	990960	.43	314247	10.64	685753	21
40	305819	10.19	990934	.44	314885	10.62	685115	20
41	9.306430	10.17	9.990908	.44	9.315523	10.61	10.684477	19
42	307041	10.16	990882	.44	316159	10.60	683841	18
43	307650	10.14	990855	.44	316795	10.58	683205	17
44	308259	10.13	990829	.44	317430	10.57	682570	16
45	308867	10.11	990803	.44	318064	10.55	681936	15
46	309474	10.10	990777	.44	318697	10.54	681303	14
47	310080	10.08	990750	.44	319329	10.53	680671	13
48	310685	10.07	990724	.44	319961	10.51	680039	12
49	311289	10.05	990697	.44	320592	10.50	679408	11
50	311893	10.04	990671	.44	321222	10.48	678778	10
51	9.312495	10.03	9.990644	.44	9.321851	10.47	10.678149	9
52	313097	10.01	990618	.44	322479	10.45	677521	8
53	313698	10.00	990591	.44	323106	10.44	676894	7
54	314297	9.98	990565	.44	323733	10.43	676267	6
55	314897	9.97	990538	.44	324358	10.41	675642	5
56	315495	9.96	990511	.45	324983	10.40	675017	4
57	316092	9.94	990485	.45	325607	10.39	674393	3
58	316689	9.93	990458	.45	326231	10.37	673769	2
59	317284	9.91	990431	.45	326853	10.36	673147	1
60	317879	9.90	990404	.45	327475	10.35	672525	0
	Cosine	D.	Sine		Cotang.	D.	Tang.	M.



M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9.317879	9.90	9.990404	.45	9.327474	10.35	10.672526	60
1	318473	9.88	990378	.45	328095	10.33	671905	59
2	319066	9.87	990351	.45	328715	10.32	671285	58
3	319658	9.86	990324	.45	329334	10.30	670666	57
4	320249	9.84	990297	.45	329953	10.29	670047	56
5	320840	9.83	990270	.45	330570	10.28	669430	55
6	321430	9.82	990243	.45	331187	10.26	668813	54
7	322019	9.80	990215	.45	331803	10.25	668197	53
8	322607	9.79	990188	.45	332418	10.24	667582	52
9	323194	9.77	990161	.45	333033	10.23	666967	51
10	323780	9.76	990134	.45	333646	10.21	666354	50
11	9 324366	9.75	9.990107	.46	9.334259	10.20	10.665741	49
12	324950	9.73	990079	.46	334871	10.19	665129	48
13	325534	9.72	990052	.46	335482	10.17	664518	47
14	326117	9.70	990025	.46	336093	10.16	663907	46
15	326700	9.69	989997	.46	336702	10.15	663298	45
16	327281	9.68	989970	.46	337311	10.13	662689	44
17	327862	9.66	989942	.46	337919	10.12	662081	43
18	328442	9.65	989915	.46	338527	10.11	661473	42
19	329021	9.64	989887	.46	339133	10.10	660867	41
20	329599	9.62	989860	.46	339739	10.08	660261	40
21	9.330176	9.61	9.989832	.46	9.340344	10.07	10.659656	39
22	330753	9.60	989804	.46	340948	10.06	659052	38
23	331329	9.58	989777	.46	341552	10.04	658448	37
24	331903	9.57	989749	.47	342155	10.03	657845	36
25	332478	9.56	989721	.47	342757	10.02	657243	35
26	333051	9.54	989693	.47	343358	10.00	656642	34
27	333624	9.53	989665	.47	343958	9.99	656042	33
28	334195	9.52	989637	.47	344558	9.98	655442	32
29	334766	9.50	989609	.47	345157	9.97	654843	31
30	335337	9.49	989582	.47	345755	9.96	654245	30
31	9.335906	9.48	9.989553	.47	9.346353	9.94	10.653647	29
32	336475	9.46	989525	.47	346949	9.93	653051	28
33	337043	9.45	989497	.47	347545	9.92	652455	27
34	337610	9.44	989469	.47	348141	9.91	651859	26
35	338176	9.43	989441	.47	348735	9.90	651265	25
36	338742	9.41	989413	.47	349329	9.88	650671	24
37	339306	9.40	989384	.47	349922	9.87	650078	23
38	339871	9.39	989356	.47	350514	9.86	649486	22
39	340434	9.37	989328	.47	351106	9.85	648894	21
40	340996	9.36	989300	.47	351697	9.83	648303	20
41	9.341558	9.35	9.989271	.47	9.352287	9.82	10.647713	19
42	342119	9.34	989243	.47	352876	9.81	647124	18
43	342679	9.32	989214	.47	353465	9.80	646535	17
44	343239	9.31	989186	.47	354053	9.79	645947	16
45	343797	9.30	989157	.47	354640	9.77	645360	15
46	344355	9.29	989128	.48	355227	9.76	644773	14
47	344912	9.27	989100	.48	355813	9.75	644187	13
48	345469	9.26	989071	.48	356398	9.74	643602	12
49	346024	9.25	989042	.48	356982	9.73	643018	11
50	346579	9.24	989014	.48	357566	9.71	642434	10
51	9.347134	9.22	9.988985	.48	9.358149	9.70	10.641851	9
52	347687	9.21	988956	.48	358731	9.69	641269	8
53	348240	9.20	988927	.48	359313	9.68	640687	7
54	348792	9.19	988898	.48	359893	9.67	640107	6
55	349343	9.17	988869	.48	360474	9.66	639526	5
56	349893	9.16	988840	.48	361053	9.65	638947	4
57	350443	9.15	988811	.49	361632	9.63	638368	3
58	350992	9.14	988782	.49	362210	9.62	637790	2
59	351540	9.13	988753	.49	362787	9.61	637213	1
60	352088	9.11	988724	.49	363364	9.60	636636	0
	Cosine	D.	Sine		Cotang.	D.	Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9.352088	9.11	9.988724	.49	9.363364	9.60	10.636636	60
1	352635	9.10	988695	.49	363940	9.59	636060	59
2	353181	9.09	988666	.49	364515	9.58	635485	58
3	353726	9.08	988636	.49	365090	9.57	634910	57
4	354271	9.07	988607	.49	365664	9.56	634336	56
5	354815	9.05	988578	.49	366237	9.55	633763	55
6	355358	9.04	988548	.49	366810	9.53	633190	54
7	355901	9.03	988519	.49	367382	9.52	632618	53
8	356443	9.02	988489	.49	367953	9.51	632047	52
9	356984	9.01	988460	.49	368524	9.50	631476	51
10	357524	8.99	988430	.49	369094	9.49	630906	50
11	9.358064	8.98	9.988401	.49	9.369663	9.48	10.630337	49
12	358603	8.97	988371	.49	370232	9.46	629768	48
13	359141	8.96	988342	.49	370799	9.45	629201	47
14	359678	8.95	988312	.50	371367	9.44	628633	46
15	360215	8.93	988282	.50	371933	9.43	628067	45
16	360752	8.92	988252	.50	372499	9.42	627501	44
17	361287	8.91	988223	.50	373064	9.41	626936	43
18	361822	8.90	988193	.50	373629	9.40	626371	42
19	362356	8.89	988163	.50	374193	9.39	625807	41
20	362889	8.88	988133	.50	374756	9.38	625244	40
21	9.363422	8.87	9.988103	.50	9.375319	9.37	10.624681	39
22	363954	8.85	988073	.50	375881	9.35	624119	38
23	364485	8.84	988043	.50	376442	9.34	623558	37
24	365016	8.83	988013	.50	377003	9.33	622997	36
25	365546	8.82	987983	.50	377563	9.32	622437	35
26	366075	8.81	987953	.50	378122	9.31	621878	34
27	366604	8.80	987922	.50	378681	9.30	621319	33
28	367131	8.79	987892	.50	379239	9.29	620761	32
29	367659	8.77	987862	.50	379797	9.28	620203	31
30	368185	8.76	987832	.51	380354	9.27	619646	30
31	9.368711	8.75	9.987801	.51	9.380910	9.26	10.619090	29
32	369236	8.74	987771	.51	381466	9.25	618534	28
33	369761	8.73	987740	.51	382020	9.24	617980	27
34	370285	8.72	987710	.51	382575	9.23	617425	26
35	370808	8.71	987679	.51	383129	9.22	616871	25
36	371330	8.70	987649	.51	383682	9.21	616318	24
37	371852	8.69	987618	.51	384234	9.20	615766	23
38	372373	8.67	987588	.51	384786	9.19	615214	22
39	372894	8.66	987557	.51	385337	9.18	614663	21
40	373414	8.65	987526	.51	385888	9.17	614112	20
41	9.373933	8.64	9.987496	.51	9.386438	9.15	10.613562	19
42	374452	8.63	987465	.51	386987	9.14	613013	18
43	374970	8.62	987434	.51	387536	9.13	612464	17
44	375487	8.61	987403	.52	388084	9.12	611916	16
45	376003	8.60	987372	.52	388631	9.11	611369	15
46	376519	8.59	987341	.52	389178	9.10	610822	14
47	377035	8.58	987310	.52	389724	9.09	610276	13
48	377549	8.57	987279	.52	390270	9.08	609730	12
49	378063	8.56	987248	.52	390815	9.07	609185	11
50	378577	8.54	987217	.52	391360	9.06	608640	10
51	9.379089	8.53	9.987186	.52	9.391903	9.05	10.608097	9
52	379601	8.52	987155	.52	392447	9.04	607553	8
53	380113	8.51	987124	.52	392989	9.03	607011	7
54	380624	8.50	987092	.52	393531	9.02	606469	6
55	381134	8.49	987061	.52	394073	9.01	605927	5
56	381643	8.48	987030	.52	394614	9.00	605386	4
57	382152	8.47	986998	.52	395154	8.99	604846	3
58	382661	8.46	986967	.52	395694	8.98	604306	2
59	383168	8.45	986936	.52	396233	8.97	603767	1
60	383675	8.44	986904	.52	396771	8.96	603229	0
	Cosine	D.	Sine		Cotang.	D.	Tang.	M.



M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9.383675	8.44	9.986904	.52	9.396771	8.96	10.603229	60
1	384182	8.43	986873	.53	397309	8.96	602691	59
2	384687	8.42	986841	.53	397846	8.95	602154	58
3	385192	8.41	986809	.53	398383	8.94	601617	57
4	385697	8.40	986778	.53	398919	8.93	601081	56
5	386201	8.39	986746	.53	399455	8.92	600545	55
6	386704	8.38	986714	.53	399990	8.91	600010	54
7	387207	8.37	986683	.53	400524	8.90	599476	53
8	387709	8.36	986651	.53	401058	8.89	598942	52
9	388210	8.35	986619	.53	401591	8.88	598409	51
10	388711	8.34	986587	.53	402124	8.87	597876	50
11	9.389211	8.33	9.986555	.53	9.402656	8.86	10.597344	49
12	389711	8.32	986523	.53	403187	8.85	596813	48
13	390210	8.31	986491	.53	403718	8.84	596282	47
14	390708	8.30	986459	.53	404249	8.83	595751	46
15	391206	8.28	986427	.53	404778	8.82	595222	45
16	391703	8.27	986395	.53	405308	8.81	594692	44
17	392199	8.26	986363	.54	405836	8.80	594164	43
18	392695	8.25	986331	.54	406364	8.79	593636	42
19	393191	8.24	986299	.54	406892	8.78	593108	41
20	393685	8.23	986266	.54	407419	8.77	592581	40
21	9.394179	8.22	9.986234	.54	9.407945	8.76	10.592055	39
22	394673	8.21	986202	.54	408471	8.75	591529	38
23	395166	8.20	986169	.54	408997	8.74	591003	37
24	395658	8.19	986137	.54	409521	8.74	590479	36
25	396150	8.18	986104	.54	410045	8.73	589955	35
26	396641	8.17	986072	.54	410569	8.72	589431	34
27	397132	8.17	986039	.54	411092	8.71	588908	33
28	397621	8.16	986007	.54	411615	8.70	588385	32
29	398111	8.15	985974	.54	412137	8.69	587863	31
30	398600	8.14	985942	.54	412658	8.68	587342	30
31	9.399088	8.13	9.985909	.55	9.413179	8.67	10.586821	29
32	399575	8.12	985876	.55	413699	8.66	586301	28
33	400062	8.11	985843	.55	414219	8.65	585781	27
34	400549	8.10	985811	.55	414738	8.64	585262	26
35	401035	8.09	985778	.55	415257	8.64	584743	25
36	401520	8.08	985745	.55	415775	8.63	584225	24
37	402005	8.07	985712	.55	416293	8.62	583707	23
38	402489	8.06	985679	.55	416810	8.61	583190	22
39	402972	8.05	985646	.55	417326	8.60	582674	21
40	403455	8.04	985613	.55	417842	8.59	582158	20
41	9.403938	8.03	9.985580	.55	9.418358	8.58	10.581642	19
42	404420	8.02	985547	.55	418873	8.57	581127	18
43	404901	8.01	985514	.55	419387	8.56	580613	17
44	405382	8.00	985480	.55	419901	8.55	580099	16
45	405862	7.99	985447	.55	420415	8.55	579585	15
46	406341	7.98	985414	.56	420927	8.54	579073	14
47	406820	7.97	985380	.56	421440	8.53	578560	13
48	407299	7.96	985347	.56	421952	8.52	578048	12
49	407777	7.95	985314	.56	422463	8.51	577537	11
50	408254	7.94	985280	.56	422974	8.50	577026	10
51	9.408731	7.94	9.985247	.56	9.423484	8.49	10.576516	9
52	409207	7.93	985213	.56	423993	8.48	576007	8
53	409682	7.92	985180	.56	424503	8.48	575497	7
54	410157	7.91	985146	.56	425011	8.47	574989	6
55	410632	7.90	985113	.56	425519	8.46	574481	5
56	411106	7.89	985079	.56	426027	8.45	573973	4
57	411579	7.88	985045	.56	426534	8.44	573466	3
58	412052	7.87	985011	.56	427041	8.43	572959	2
59	412524	7.86	984978	.56	427547	8.43	572453	1
60	412996	7.85	984944	.56	428052	8.42	571948	0
	Cosine	D.	Sine		Cotang.	D.	Tang.	M.



M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9.412996	7.85	9.984944	.57	9.428052	8.42	10.571948	60
1	413467	7.84	984910	.57	428557	8.41	571443	59
2	413938	7.83	984876	.57	429062	8.40	570938	58
3	414408	7.83	984842	.57	429566	8.39	570434	57
4	414878	7.82	984808	.57	430070	8.38	569930	56
5	415347	7.81	984774	.57	430573	8.38	569427	55
6	415815	7.80	984740	.57	431075	8.37	568925	54
7	416283	7.79	984706	.57	431577	8.36	568423	53
8	416751	7.78	984672	.57	432079	8.35	567921	52
9	417217	7.77	984637	.57	432580	8.34	567420	51
10	417684	7.76	984603	.57	433080	8.33	566920	50
11	9.418150	7.75	9.984569	.57	9.433580	8.32	10.566420	49
12	418615	7.74	984535	.57	434080	8.32	565920	48
13	419079	7.73	984500	.57	434579	8.31	565421	47
14	419544	7.73	984466	.57	435078	8.30	564922	46
15	420007	7.72	984432	.58	435576	8.29	564424	45
16	420470	7.71	984397	.58	436073	8.28	563927	44
17	420933	7.70	984363	.58	436570	8.28	563430	43
18	421395	7.69	984328	.58	437067	8.27	562933	42
19	421857	7.68	984294	.58	437563	8.26	562437	41
20	422318	7.67	984259	.58	438059	8.25	561941	40
21	9.422778	7.67	9.984224	.58	9.438554	8.24	10.561446	39
22	423238	7.66	984190	.58	439048	8.23	560952	38
23	423697	7.65	984155	.58	439543	8.23	560457	37
24	424156	7.64	984120	.58	440036	8.22	559964	36
25	424615	7.63	984085	.58	440529	8.21	559471	35
26	425073	7.62	984050	.58	441022	8.20	558978	34
27	425530	7.61	984015	.58	441514	8.19	558486	33
28	425987	7.60	983981	.58	442006	8.19	557994	32
29	426443	7.60	983946	.58	442497	8.18	557503	31
30	426899	7.59	983911	.58	442988	8.17	557012	30
31	9.427354	7.58	9.983875	.58	9.443479	8.16	10.556521	29
32	427809	7.57	983840	.59	443968	8.16	556032	28
33	428263	7.56	983805	.59	444458	8.15	555542	27
34	428717	7.55	983770	.59	444947	8.14	555053	26
35	429170	7.54	983735	.59	445435	8.13	554565	25
36	429623	7.53	983700	.59	445923	8.12	554077	24
37	430075	7.52	983664	.59	446411	8.12	553589	23
38	430527	7.52	983629	.59	446898	8.11	553102	22
39	430978	7.51	983594	.59	447384	8.10	552616	21
40	431429	7.50	983558	.59	447870	8.09	552130	20
41	9.431879	7.49	9.983523	.59	9.448356	8.09	10.551644	19
42	432329	7.49	983487	.59	448841	8.08	551159	18
43	432778	7.48	983452	.59	449326	8.07	550674	17
44	433226	7.47	983416	.59	449810	8.06	550190	16
45	433675	7.46	983381	.59	450294	8.06	549706	15
46	434122	7.45	983345	.59	450777	8.05	549223	14
47	434569	7.44	983309	.59	451260	8.04	548740	13
48	435016	7.44	983273	.60	451743	8.03	548257	12
49	435462	7.43	983238	.60	452225	8.02	547775	11
50	435908	7.42	983202	.60	452706	8.02	547294	10
51	9.436353	7.41	9.983166	.60	9.453187	8.01	10.546813	9
52	436798	7.40	983130	.60	453668	8.00	546332	8
53	437242	7.40	983094	.60	454148	7.99	545852	7
54	437686	7.39	983058	.60	454628	7.99	545372	6
55	438129	7.38	983022	.60	455107	7.98	544893	5
56	438572	7.37	982986	.60	455586	7.97	544414	4
57	439014	7.36	982950	.60	456064	7.96	543936	3
58	439456	7.36	982914	.60	456542	7.96	543458	2
59	439897	7.35	982878	.60	457019	7.95	542981	1
60	440338	7.34	982842	.60	457496	7.94	542504	0
	Cosine	D.	Sine		Cotang.	D.	Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9.440338	7.34	9.982842	.60	9.457496	7.94	10.542504	60
1	440778	7.23	982805	.60	457973	7.93	542027	59
2	441218	7.32	982769	.61	458449	7.93	541551	58
3	441658	7.31	982733	.61	458925	7.92	541075	57
4	442096	7.31	982696	.61	459400	7.91	540600	56
5	442535	7.30	982660	.61	459875	7.90	540125	55
6	442973	7.29	982624	.61	460349	7.90	539651	54
7	443410	7.28	982587	.61	460823	7.89	539177	53
8	443847	7.27	982551	.61	461297	7.88	538703	52
9	444284	7.27	982514	.61	461770	7.88	538230	51
10	444720	7.26	982477	.61	462242	7.87	537758	50
11	9.445155	7.25	9.982441	.61	9.462714	7.86	10.537286	49
12	445590	7.24	982404	.61	463186	7.85	536814	48
13	446025	7.23	982367	.61	463658	7.85	536342	47
14	446459	7.23	982331	.61	464129	7.84	535871	46
15	446893	7.22	982294	.61	464599	7.83	535401	45
16	447326	7.21	982257	.61	465069	7.83	534931	44
17	447759	7.20	982220	.62	465539	7.82	534461	43
18	448191	7.20	982183	.62	466008	7.81	533992	42
19	448623	7.19	982146	.62	466476	7.80	533524	41
20	449054	7.18	982109	.62	466945	7.80	533055	40
21	9.449485	7.17	9.982072	.62	9.467413	7.79	10.532587	39
22	449915	7.16	982035	.62	467880	7.78	532120	38
23	450345	7.16	981998	.62	468347	7.78	531653	37
24	450775	7.15	981961	.62	468814	7.77	531186	36
25	451204	7.14	981924	.62	469280	7.76	530720	35
26	451632	7.13	981886	.62	469746	7.75	530254	34
27	452060	7.13	981849	.62	470211	7.75	529789	33
28	452488	7.12	981812	.62	470676	7.74	529324	32
29	452915	7.11	981774	.62	471141	7.73	528859	31
30	453342	7.10	981737	.62	471605	7.73	528395	30
31	9.453768	7.10	9.981699	.63	9.472068	7.72	10.527932	29
32	454194	7.09	981662	.63	472532	7.71	527468	28
33	454619	7.08	981625	.63	472995	7.71	527005	27
34	455044	7.07	981587	.63	473457	7.70	526543	26
35	455469	7.07	981549	.63	473919	7.69	526081	25
36	455893	7.06	981512	.63	474381	7.69	525619	24
37	456316	7.05	981474	.63	474842	7.68	525158	23
38	456739	7.04	981436	.63	475303	7.67	524697	22
39	457162	7.04	981399	.63	475763	7.67	524237	21
40	457584	7.03	981361	.63	476223	7.66	523777	20
41	9.458006	7.02	9.981323	.63	9.476683	7.65	10.523317	19
42	458427	7.01	981285	.63	477142	7.65	522858	18
43	458848	7.01	981247	.63	477601	7.64	522399	17
44	459268	7.00	981209	.63	478059	7.63	521941	16
45	459688	6.99	981171	.63	478517	7.63	521483	15
46	460108	6.98	981133	.64	478975	7.62	521025	14
47	460527	6.98	981095	.64	479432	7.61	520568	13
48	460946	6.97	981057	.64	479889	7.61	520111	12
49	461364	6.96	981019	.64	480345	7.60	519655	11
50	461782	6.95	980981	.64	480801	7.59	519199	10
51	9.462199	6.95	9.980942	.64	9.481257	7.59	10.518743	9
52	462616	6.94	980904	.64	481712	7.58	518288	8
53	463032	6.93	980866	.64	482167	7.57	517833	7
54	463448	6.93	980827	.64	482621	7.57	517379	6
55	463864	6.92	980789	.64	483075	7.56	516925	5
56	464279	6.91	980750	.64	483529	7.55	516471	4
57	464694	6.90	980712	.64	483982	7.55	516018	3
58	465108	6.90	980673	.64	484435	7.54	515565	2
59	465522	6.89	980635	.64	484887	7.53	515113	1
60	465935	6.88	980596	.64	485339	7.53	514661	0
	Cosine	D.	Sine		Cotang.	D.	Tang.	M.



M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9.465935	6.88	9.980596	.64	9.485339	7.55	10.514661	60
1	466348	6.88	980558	.64	485791	7.52	514209	59
2	466761	6.87	980519	.65	486242	7.51	513758	58
3	467173	6.86	980480	.65	486693	7.51	513307	57
4	467585	6.85	980442	.65	487143	7.50	512857	56
5	467996	6.85	980403	.65	487593	7.49	512407	55
6	468407	6.84	980364	.65	488043	7.49	511957	54
7	468817	6.83	980325	.65	488492	7.48	511508	53
8	469227	6.83	980286	.65	488941	7.47	511059	52
9	469637	6.81	980247	.65	489390	7.47	510610	51
10	470046	6.81	980208	.65	489838	7.46	510162	50
11	9.470455	6.81	9.980169	.65	9.490286	7.46	10.509714	49
12	470863	6.80	980130	.65	490733	7.45	509267	48
13	471271	6.79	980091	.65	491180	7.44	508820	47
14	471679	6.78	980052	.65	491627	7.44	508373	46
15	472086	6.78	980012	.65	492073	7.43	507927	45
16	472492	6.77	979973	.65	492519	7.43	507481	44
17	472898	6.76	979934	.66	492965	7.42	507035	43
18	473304	6.76	979895	.66	493410	7.41	506590	42
19	473710	6.75	979855	.66	493854	7.40	506146	41
20	474115	6.74	979816	.66	494299	7.40	505701	40
21	9.474519	6.74	9.979776	.66	9.494743	7.40	10.505257	39
22	474923	6.73	979737	.66	495186	7.39	504814	38
23	475327	6.72	979697	.66	495630	7.38	504370	37
24	475730	6.72	979658	.66	496073	7.37	503927	36
25	476133	6.71	979618	.66	496515	7.37	503485	35
26	476536	6.70	979579	.66	496957	7.36	503043	34
27	476938	6.69	979539	.66	497399	7.36	502601	33
28	477340	6.69	979499	.66	497841	7.35	502159	32
29	477741	6.68	979459	.66	498282	7.34	501718	31
30	478142	6.67	979420	.66	498722	7.34	501278	30
31	9.478542	6.67	9.979380	.66	9.499163	7.33	10.500837	29
32	478942	6.66	979340	.66	499603	7.33	500397	28
33	479342	6.65	979300	.67	500042	7.32	499958	27
34	479741	6.65	979260	.67	500481	7.31	499519	26
35	480140	6.64	979220	.67	500920	7.31	499080	25
36	480539	6.63	979180	.67	501359	7.30	498641	24
37	480937	6.63	979140	.67	501797	7.30	498203	23
38	481334	6.62	979100	.67	502235	7.29	497765	22
39	481731	6.61	979059	.67	502672	7.28	497328	21
40	482128	6.61	979019	.67	503109	7.28	496891	20
41	9.482525	6.60	9.978979	.67	9.503546	7.27	10.496454	19
42	482921	6.59	978939	.67	503982	7.27	496018	18
43	483316	6.59	978898	.67	504418	7.26	495582	17
44	483712	6.58	978858	.67	504854	7.25	495146	16
45	484107	6.57	978817	.67	505289	7.25	494711	15
46	484501	6.57	978777	.67	505724	7.24	494276	14
47	484895	6.56	978736	.67	506159	7.24	493841	13
48	485289	6.55	978696	.68	506593	7.23	493407	12
49	485682	6.55	978655	.68	507027	7.22	492973	11
50	486075	6.54	978615	.68	507460	7.22	492540	10
51	9.486467	6.53	9.978574	.68	9.507893	7.21	10.492107	9
52	486860	6.53	978533	.68	508326	7.21	491674	8
53	487251	6.52	978493	.68	508759	7.20	491241	7
54	487643	6.51	978452	.68	509191	7.19	490809	6
55	488034	6.51	978411	.68	509622	7.19	490378	5
56	488424	6.50	978370	.68	510054	7.18	489946	4
57	488814	6.50	978329	.68	510485	7.18	489515	3
58	489204	6.49	978288	.68	510916	7.17	489084	2
59	489593	6.48	978247	.68	511346	7.16	488654	1
60	489982	6.48	978206	.68	511776	7.16	488224	0
	Cosine	D.	Sine	D.	Cotang.	D.	Tang.	M.



M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9.489982	6.48	9.978206	.68	9.511776	7.16	10.488224	60
1	490371	6.48	978165	.68	512206	7.16	487794	59
2	490759	6.47	978124	.68	512635	7.15	487365	58
3	491147	6.46	978083	.69	513064	7.14	486936	57
4	491535	6.46	978042	.69	513493	7.14	486507	56
5	491922	6.45	978001	.69	513921	7.13	486079	55
6	492308	6.44	977959	.69	514349	7.13	485651	54
7	492695	6.44	977918	.69	514777	7.12	485223	53
8	493081	6.43	977877	.69	515204	7.12	484796	52
9	493466	6.42	977835	.69	515631	7.11	484369	51
10	493851	6.42	977794	.69	516057	7.10	483943	50
11	9.494236	6.41	9.977752	.69	9.516484	7.10	10.483516	49
12	494621	6.41	977711	.69	516910	7.09	483090	48
13	495005	6.40	977669	.69	517335	7.09	482665	47
14	495388	6.39	977628	.69	517761	7.08	482239	46
15	495772	6.39	977586	.69	518185	7.08	481815	45
16	496154	6.38	977544	.70	518610	7.07	481390	44
17	496537	6.37	977503	.70	519034	7.06	480966	43
18	496919	6.37	977461	.70	519458	7.06	480542	42
19	497301	6.36	977419	.70	519882	7.05	480118	41
20	497682	6.36	977377	.70	520305	7.05	479695	40
21	9.498064	6.35	9.977335	.70	9.520728	7.04	10.479272	39
22	498444	6.34	977293	.70	521151	7.03	478849	38
23	498825	6.34	977251	.70	521573	7.03	478427	37
24	499204	6.33	977209	.70	521995	7.03	478005	36
25	499584	6.32	977167	.70	522417	7.02	477583	35
26	499963	6.32	977125	.70	522838	7.02	477162	34
27	500342	6.31	977083	.70	523259	7.01	476741	33
28	500721	6.31	977041	.70	523680	7.01	476320	32
29	501099	6.30	976999	.70	524100	7.00	475900	31
30	501476	6.29	976957	.70	524520	6.99	475480	30
31	9.501854	6.29	9.976914	.70	9.524939	6.99	10.475061	29
32	502231	6.28	976872	.71	525359	6.98	474641	28
33	502607	6.28	976830	.71	525778	6.98	474222	27
34	502984	6.27	976787	.71	526197	6.97	473803	26
35	503360	6.26	976745	.71	526615	6.97	473385	25
36	503735	6.26	976702	.71	527033	6.96	472967	24
37	504110	6.25	976660	.71	527451	6.96	472549	23
38	504485	6.25	976617	.71	527868	6.95	472132	22
39	504860	6.24	976574	.71	528285	6.95	471715	21
40	505234	6.23	976532	.71	528702	6.94	471298	20
41	9.505608	6.23	9.976489	.71	9.529119	6.93	10.470881	19
42	505981	6.22	976446	.71	529535	6.93	470465	18
43	506354	6.22	976404	.71	529950	6.93	470050	17
44	506727	6.21	976361	.71	530366	6.92	469634	16
45	507099	6.20	976318	.71	530781	6.91	469219	15
46	507471	6.20	976275	.71	531196	6.91	468804	14
47	507843	6.19	976232	.72	531611	6.90	468389	13
48	508214	6.19	976189	.72	532025	6.90	467975	12
49	508585	6.18	976146	.72	532439	6.89	467561	11
50	508956	6.18	976103	.72	532853	6.89	467147	10
51	9.509326	6.17	9.976060	.72	9.533266	6.88	10.466734	9
52	509696	6.16	976017	.72	533679	6.88	466321	8
53	510065	6.16	975974	.72	534092	6.87	465908	7
54	510434	6.15	975930	.72	534504	6.87	465496	6
55	510803	6.15	975887	.72	534916	6.86	465084	5
56	511172	6.14	975844	.72	535328	6.86	464672	4
57	511540	6.13	975800	.72	535739	6.85	464261	3
58	511907	6.13	975757	.72	536150	6.85	463850	2
59	512275	6.12	975714	.72	536561	6.84	463439	1
60	512642	6.12	975670	.72	536972	6.84	463028	0
	Cosine	D.	Sine	D.	Cotang.	D.	Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
c	9.512642	6.12	9.975670	.73	9.536972	6.84	10.463028	60
1	513009	6.11	975627	.73	537382	6.83	462618	59
2	513375	6.11	975583	.73	537792	6.83	462208	58
3	513741	6.10	975539	.73	538202	6.82	461798	57
4	514107	6.09	975496	.73	538611	6.82	461389	56
5	514472	6.09	975452	.73	539020	6.81	460980	55
6	514837	6.08	975408	.73	539429	6.81	460571	54
7	515202	6.08	975365	.73	539837	6.80	460163	53
8	515566	6.07	975321	.73	540245	6.80	459755	52
9	515930	6.07	975277	.73	540653	6.79	459347	51
10	516294	6.06	975233	.73	541061	6.79	458939	50
11	9.516657	6.05	9.975189	.73	9.541468	6.78	10.458532	49
12	517020	6.05	975145	.73	541875	6.78	458125	48
13	517382	6.04	975101	.73	542281	6.77	457719	47
14	517745	6.04	975057	.73	542688	6.77	457312	46
15	518107	6.03	975013	.73	543094	6.76	456906	45
16	518468	6.03	974969	.74	543499	6.76	456501	44
17	518829	6.02	974925	.74	543905	6.75	456095	43
18	519190	6.01	974880	.74	544310	6.75	455690	42
19	519551	6.01	974836	.74	544715	6.74	455285	41
20	519911	6.00	974792	.74	545119	6.74	454881	40
21	9.520271	6.00	9.974748	.74	9.545524	6.73	10.454476	39
22	520631	5.99	974703	.74	545928	6.73	454072	38
23	520990	5.99	974659	.74	546331	6.72	453669	37
24	521349	5.98	974614	.74	546735	6.72	453265	36
25	521707	5.98	974570	.74	547138	6.71	452862	35
26	522066	5.97	974525	.74	547540	6.71	452460	34
27	522424	5.96	974481	.74	547943	6.70	452057	33
28	522781	5.96	974436	.74	548345	6.70	451655	32
29	523138	5.95	974391	.74	548747	6.69	451253	31
30	523495	5.95	974347	.75	549149	6.69	450851	30
31	9.523852	5.94	9.974302	.75	9.549550	6.68	10.450450	29
32	524208	5.94	974257	.75	549951	6.68	450049	28
33	524564	5.93	974212	.75	550352	6.67	449648	27
34	524920	5.93	974167	.75	550752	6.67	449248	26
35	525275	5.92	974122	.75	551152	6.66	448848	25
36	525630	5.91	974077	.75	551552	6.66	448448	24
37	525984	5.91	974032	.75	551952	6.65	448048	23
38	526339	5.90	973987	.75	552351	6.65	447649	22
39	526693	5.90	973942	.75	552750	6.65	447250	21
40	527046	5.89	973897	.75	553149	6.64	446851	20
41	9.527400	5.89	9.973852	.75	9.553548	6.64	10.446452	19
42	527753	5.88	973807	.75	553946	6.63	446054	18
43	528105	5.88	973761	.75	554344	6.63	445656	17
44	528458	5.87	973716	.76	554741	6.62	445259	16
45	528810	5.87	973671	.76	555139	6.62	444861	15
46	529161	5.86	973625	.76	555536	6.61	444464	14
47	529513	5.86	973580	.76	555933	6.61	444067	13
48	529864	5.85	973535	.76	556329	6.60	443671	12
49	530215	5.85	973489	.76	556725	6.60	443275	11
50	530565	5.84	973444	.76	557121	6.59	442879	10
51	9.530915	5.84	9.973398	.76	9.557517	6.59	10.442483	9
52	531265	5.83	973352	.76	557913	6.59	442087	8
53	531614	5.82	973307	.76	558308	6.58	441692	7
54	531963	5.82	973261	.76	558702	6.58	441298	6
55	532312	5.81	973215	.76	559097	6.57	440903	5
56	532661	5.81	973169	.76	559491	6.57	440509	4
57	533009	5.80	973124	.76	559885	6.56	440115	3
58	533357	5.80	973078	.76	560279	6.56	439721	2
59	533704	5.79	973032	.77	560673	6.55	439327	1
60	534052	5.78	972986	.77	561066	6.55	438934	0
	Cosine	D.	Sine	D.	Cotang.	D.	Tang.	M.



M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9.534052	5.78	9.972986	.77	9.561066	6.55	10.438934	60
1	534399	5.77	972940	.77	561459	6.54	438541	59
2	534745	5.77	972894	.77	561851	6.54	438149	58
3	535092	5.77	972848	.77	562244	6.53	437756	57
4	535438	5.76	972802	.77	562636	6.53	437364	56
5	535783	5.76	972755	.77	563028	6.53	436972	55
6	536129	5.75	972709	.77	563419	6.52	436581	54
7	536474	5.74	972663	.77	563811	6.52	436189	53
8	536818	5.74	972617	.77	564202	6.51	435798	52
9	537163	5.73	972570	.77	564592	6.51	435408	51
10	537507	5.73	972524	.77	564983	6.50	435017	50
11	9.537851	5.72	9.972478	.77	9.565373	6.50	10.434627	49
12	538194	5.72	972431	.78	565763	6.49	434237	48
13	538538	5.71	972385	.78	566153	6.49	433847	47
14	538880	5.71	972338	.78	566542	6.49	433458	46
15	539223	5.70	972291	.78	566932	6.48	433068	45
16	539565	5.70	972245	.78	567320	6.48	432680	44
17	539907	5.69	972198	.78	567709	6.47	432291	43
18	540249	5.69	972151	.78	568098	6.47	431902	42
19	540590	5.68	972105	.78	568486	6.46	431514	41
20	540931	5.68	972058	.78	568873	6.46	431127	40
21	9.541272	5.67	9.972011	.78	9.569261	6.45	10.430739	39
22	541613	5.67	971964	.78	569648	6.45	430352	38
23	541953	5.66	971917	.78	570035	6.45	429965	37
24	542293	5.66	971870	.78	570422	6.44	429578	36
25	542632	5.65	971823	.78	570809	6.44	429191	35
26	542971	5.65	971776	.78	571195	6.43	428805	34
27	543310	5.64	971729	.79	571581	6.43	428419	33
28	543649	5.64	971682	.79	571967	6.42	428033	32
29	543987	5.63	971635	.79	572352	6.42	427648	31
30	544325	5.63	971588	.79	572738	6.42	427262	30
31	9.544663	5.62	9.971540	.79	9.573123	6.41	10.426877	29
32	545000	5.62	971493	.79	573507	6.41	426493	28
33	545338	5.61	971446	.79	573892	6.40	426108	27
34	545674	5.61	971398	.79	574276	6.40	425724	26
35	546011	5.60	971351	.79	574660	6.39	425340	25
36	546347	5.60	971303	.79	575044	6.39	424956	24
37	546683	5.59	971256	.79	575427	6.39	424573	23
38	547019	5.59	971208	.79	575810	6.38	424190	22
39	547354	5.58	971161	.79	576193	6.38	423807	21
40	547689	5.58	971113	.79	576576	6.37	423424	20
41	9.548024	5.57	9.971066	.80	9.576958	6.37	10.423041	19
42	548359	5.57	971018	.80	577341	6.36	422659	18
43	548693	5.56	970970	.80	577723	6.36	422277	17
44	549027	5.56	970922	.80	578104	6.36	421896	16
45	549360	5.55	970874	.80	578486	6.35	421514	15
46	549693	5.55	970827	.80	578867	6.35	421133	14
47	550026	5.54	970779	.80	579248	6.34	420752	13
48	550359	5.54	970731	.80	579629	6.34	420371	12
49	550692	5.53	970683	.80	580009	6.34	419991	11
50	551024	5.53	970635	.80	580389	6.33	419611	10
51	9.551356	5.52	9.970586	.80	9.580769	6.33	10.419231	9
52	551687	5.52	970538	.80	581149	6.32	418851	8
53	552018	5.52	970490	.80	581528	6.32	418472	7
54	552349	5.51	970442	.80	581907	6.32	418093	6
55	552680	5.51	970394	.80	582286	6.31	417714	5
56	553010	5.50	970345	.81	582665	6.31	417335	4
57	553341	5.50	970297	.81	583043	6.30	416957	3
58	553670	5.49	970249	.81	583422	6.30	416578	2
59	554000	5.49	970200	.81	583800	6.29	416200	1
60	554329	5.48	970152	.81	584177	6.29	415823	0
	Cosine	D.	Sine	D.	Cotang.	D.	Tang.	M.



M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9.554329	5.48	9.970152	.81	9.584177	6.29	10.415823	60
1	554658	5.48	970103	.81	584555	6.29	415445	59
2	554987	5.47	970055	.81	584932	6.28	415068	58
3	555315	5.47	970006	.81	585309	6.28	414691	57
4	555643	5.46	969957	.81	585686	6.27	414314	56
5	555971	5.46	969909	.81	586062	6.27	413938	55
6	556299	5.45	969860	.81	586439	6.27	413561	54
7	556626	5.45	969811	.81	586815	6.26	413185	53
8	556953	5.44	969762	.81	587190	6.26	412810	52
9	557280	5.44	969714	.81	587566	6.25	412434	51
10	557606	5.43	969665	.81	587941	6.25	412059	50
11	9.557932	5.43	9.969616	.82	9.588316	6.25	10.411684	49
12	558258	5.43	969567	.82	588691	6.24	411309	48
13	558583	5.42	969518	.82	589066	6.24	410934	47
14	558909	5.42	969469	.82	589440	6.23	410560	46
15	559234	5.41	969420	.82	589814	6.23	410186	45
16	559558	5.41	969370	.82	590188	6.23	409812	44
17	559883	5.40	969321	.82	590562	6.22	409438	43
18	560207	5.40	969272	.82	590935	6.22	409065	42
19	560531	5.39	969223	.82	591308	6.22	408692	41
20	560855	5.39	969173	.82	591681	6.21	408319	40
21	9.561178	5.38	9.969124	.82	9.592054	6.21	10.407946	39
22	561501	5.38	969075	.82	592426	6.20	407574	38
23	561824	5.37	969025	.82	592798	6.20	407202	37
24	562146	5.37	968976	.82	593170	6.19	406829	36
25	562468	5.36	968926	.83	593542	6.19	406458	35
26	562790	5.36	968877	.83	593914	6.18	406086	34
27	563112	5.36	968827	.83	594285	6.18	405715	33
28	563433	5.35	968777	.83	594656	6.18	405344	32
29	563755	5.35	968728	.83	595027	6.17	404973	31
30	564075	5.34	968678	.83	595398	6.17	404602	30
31	9.564396	5.34	9.968628	.83	9.595768	6.17	10.404232	29
32	564716	5.33	968578	.83	596138	6.16	403862	28
33	565036	5.33	968528	.83	596508	6.16	403492	27
34	565356	5.32	968479	.83	596878	6.16	403122	26
35	565676	5.32	968429	.83	597247	6.15	402753	25
36	565995	5.31	968379	.83	597616	6.15	402384	24
37	566314	5.31	968329	.83	597985	6.15	402015	23
38	566632	5.31	968278	.83	598354	6.14	401646	22
39	566951	5.30	968228	.84	598722	6.14	401278	21
40	567269	5.30	968178	.84	599091	6.13	400909	20
41	9.567587	5.29	9.968128	.84	9.599459	6.13	10.400541	19
42	567904	5.29	968078	.84	599827	6.13	400173	18
43	568222	5.28	968027	.84	600194	6.12	399806	17
44	568539	5.28	967977	.84	600562	6.12	399438	16
45	568856	5.28	967927	.84	600929	6.11	399071	15
46	569172	5.27	967876	.84	601296	6.11	398704	14
47	569488	5.27	967826	.84	601662	6.11	398338	13
48	569804	5.26	967775	.84	602029	6.10	397971	12
49	570120	5.26	967725	.84	602395	6.10	397605	11
50	570435	5.25	967674	.84	602761	6.10	397239	10
51	9.570751	5.25	9.967624	.84	9.603127	6.09	10.396873	9
52	571066	5.24	967573	.84	603493	6.09	396507	8
53	571380	5.24	967522	.85	603858	6.09	396142	7
54	571695	5.23	967471	.85	604223	6.08	395777	6
55	572009	5.23	967421	.85	604588	6.08	395412	5
56	572323	5.23	967370	.85	604953	6.07	395047	4
57	572636	5.22	967319	.85	605317	6.07	394683	3
58	572950	5.22	967268	.85	605682	6.07	394318	2
59	573263	5.21	967217	.85	606046	6.06	393954	1
60	573575	5.21	967166	.85	606410	6.06	393590	0
	Cosine	D.	Sine	D.	Cotang.	D.	Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9.573575	5.21	9.967166	.85	9.606410	6.06	10.393590	60
1	573888	5.20	967115	.85	606773	6.06	393227	59
2	574200	5.20	967064	.85	607137	6.05	392863	58
3	574512	5.19	967013	.85	607500	6.05	392500	57
4	574824	5.19	966961	.85	607863	6.04	392137	56
5	575136	5.19	966910	.85	608225	6.04	391775	55
6	575447	5.18	966859	.85	608588	6.04	391412	54
7	575758	5.18	966808	.85	608950	6.03	391050	53
8	576069	5.17	966756	.86	609312	6.03	390688	52
9	576379	5.17	966705	.86	609674	6.03	390326	51
10	576689	5.16	966653	.86	610036	6.02	389964	50
11	9.576999	5.16	9.966602	.86	9.610397	6.02	10.389603	49
12	577309	5.16	966550	.86	610759	6.02	389241	48
13	577618	5.15	966499	.86	611120	6.01	388880	47
14	577927	5.15	966447	.86	611480	6.01	388520	46
15	578236	5.14	966395	.86	611841	6.01	388159	45
16	578545	5.14	966344	.86	612201	6.00	387799	44
17	578853	5.13	966292	.86	612561	6.00	387439	43
18	579162	5.13	966240	.86	612921	6.00	387079	42
19	579470	5.13	966188	.86	613281	5.99	386719	41
20	579777	5.12	966136	.86	613641	5.99	386359	40
21	9.580085	5.12	9.966085	.87	9.614000	5.98	10.386000	39
22	580392	5.11	966033	.87	614359	5.98	385641	38
23	580699	5.11	965981	.87	614718	5.98	385282	37
24	581005	5.11	965928	.87	615077	5.97	384923	36
25	581312	5.10	965876	.87	615435	5.97	384565	35
26	581618	5.10	965824	.87	615793	5.97	384207	34
27	581924	5.09	965772	.87	616151	5.96	383849	33
28	582229	5.09	965720	.87	616509	5.96	383491	32
29	582535	5.09	965668	.87	616867	5.96	383133	31
30	582840	5.08	965615	.87	617224	5.95	382776	30
31	9.583145	5.08	9.965563	.87	9.617582	5.95	10.382418	29
32	583449	5.07	965511	.87	617939	5.95	382061	28
33	583754	5.07	965458	.87	618295	5.94	381705	27
34	584058	5.06	965406	.87	618652	5.94	381348	26
35	584361	5.06	965353	.88	619008	5.94	380992	25
36	584665	5.06	965301	.88	619364	5.93	380636	24
37	584968	5.05	965248	.88	619721	5.93	380279	23
38	585272	5.05	965195	.88	620076	5.93	379924	22
39	585574	5.04	965143	.88	620432	5.92	379568	21
40	585877	5.04	965090	.88	620787	5.92	379213	20
41	9.586179	5.03	9.965037	.88	9.621142	5.92	10.378858	19
42	586482	5.03	964984	.88	621497	5.91	378503	18
43	586783	5.03	964931	.88	621852	5.91	378148	17
44	587085	5.02	964879	.88	622207	5.90	377793	16
45	587386	5.02	964826	.88	622561	5.90	377439	15
46	587688	5.01	964773	.88	622915	5.90	377085	14
47	587989	5.01	964719	.88	623269	5.89	376731	13
48	588289	5.01	964666	.89	623623	5.89	376377	12
49	588590	5.00	964613	.89	623976	5.89	376024	11
50	588890	5.00	964560	.89	624330	5.88	375670	10
51	9.589190	4.99	9.964507	.89	9.624683	5.88	10.375317	9
52	589489	4.99	964454	.89	625036	5.88	374964	8
53	589789	4.99	964400	.89	625388	5.87	374612	7
54	590088	4.98	964347	.89	625741	5.87	374259	6
55	590387	4.98	964294	.89	626093	5.87	373907	5
56	590686	4.97	964240	.89	626445	5.86	373555	4
57	590984	4.97	964187	.89	626797	5.86	373203	3
58	591282	4.97	964133	.89	627149	5.86	372851	2
59	591580	4.96	964080	.89	627501	5.85	372499	1
60	591878	4.96	964026	.89	627852	5.85	372148	0
	Cosine	D.	Sine	D.	Cotang.	D.	Tang.	M.



M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9.591878	4.96	9.964026	.89	9.627852	5.85	10.372148	60
1	592176	4.95	963972	.89	628203	5.85	371797	59
2	592473	4.95	963919	.89	628554	5.85	371446	58
3	592770	4.95	963865	.90	628905	5.84	371095	57
4	593067	4.94	963811	.90	629255	5.84	370745	56
5	593363	4.94	963757	.90	629606	5.83	370394	55
6	593659	4.93	963704	.90	629956	5.83	370044	54
7	593955	4.93	963650	.90	630306	5.83	369694	53
8	594251	4.93	963596	.90	630656	5.83	369344	52
9	594547	4.92	963542	.90	631005	5.82	368995	51
10	594842	4.92	963488	.90	631355	5.82	368645	50
11	9.595137	4.91	9.963434	.90	9.631704	5.82	10.368296	49
12	595432	4.91	963379	.90	632053	5.81	367947	48
13	595727	4.91	963325	.90	632401	5.81	367599	47
14	596021	4.90	963271	.90	632750	5.81	367250	46
15	596315	4.90	963217	.90	633098	5.80	366902	45
16	596609	4.89	963163	.90	633447	5.80	366553	44
17	596903	4.89	963108	.91	633795	5.80	366205	43
18	597196	4.89	963054	.91	634143	5.79	365857	42
19	597490	4.88	962999	.91	634490	5.79	365510	41
20	597783	4.88	962945	.91	634838	5.79	365162	40
21	9.598075	4.87	9.962890	.91	9.635185	5.78	10.364815	39
22	598368	4.87	962836	.91	635532	5.78	364468	38
23	598660	4.87	962781	.91	635879	5.78	364121	37
24	598952	4.86	962727	.91	636226	5.77	363774	36
25	599244	4.86	962672	.91	636572	5.77	363428	35
26	599536	4.85	962617	.91	636919	5.77	363081	34
27	599827	4.85	962562	.91	637265	5.77	362735	33
28	600118	4.85	962508	.91	637611	5.76	362389	32
29	600409	4.84	962453	.91	637956	5.76	362044	31
30	600700	4.84	962398	.92	638302	5.76	361698	30
31	9.600990	4.84	9.962343	.92	9.638647	5.75	10.361353	29
32	601280	4.83	962288	.92	638992	5.75	361008	28
33	601570	4.83	962233	.92	639337	5.75	360663	27
34	601860	4.82	962178	.92	639682	5.74	360318	26
35	602150	4.82	962123	.92	640027	5.74	359973	25
36	602439	4.82	962067	.92	640371	5.74	359629	24
37	602728	4.81	962012	.92	640716	5.73	359284	23
38	603017	4.81	961957	.92	641060	5.73	358940	22
39	603305	4.81	961902	.92	641404	5.73	358596	21
40	603594	4.80	961846	.92	641747	5.72	358253	20
41	9.603882	4.80	9.961791	.92	9.642091	5.72	10.357909	19
42	604170	4.79	961735	.92	642434	5.72	357566	18
43	604457	4.79	961680	.92	642777	5.72	357223	17
44	604745	4.79	961624	.93	643120	5.71	356880	16
45	605032	4.78	961569	.93	643463	5.71	356537	15
46	605319	4.78	961513	.93	643806	5.71	356194	14
47	605606	4.78	961458	.93	644148	5.70	355852	13
48	605892	4.77	961402	.93	644490	5.70	355510	12
49	606179	4.77	961346	.93	644832	5.70	355168	11
50	606465	4.76	961290	.93	645174	5.69	354826	10
51	9.606751	4.76	9.961235	.93	9.645516	5.69	10.354484	9
52	607036	4.76	961179	.93	645857	5.69	354143	8
53	607322	4.75	961123	.93	646199	5.69	353801	7
54	607607	4.75	961067	.93	646540	5.68	353460	6
55	607892	4.74	961011	.93	646881	5.68	353119	5
56	608177	4.74	960955	.93	647222	5.68	352778	4
57	608461	4.74	960899	.93	647562	5.67	352438	3
58	608745	4.73	960843	.94	647903	5.67	352097	2
59	609029	4.73	960786	.94	648243	5.67	351757	1
60	609313	4.73	960730	.94	648583	5.66	351417	0
	Cosine	D.	Sine	D.	Cotang.	D.	Tang.	M.



M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9.609313	4.73	9.960730	.94	9.648583	5.66	10.351417	60
1	609597	4.72	960674	.94	648923	5.66	351077	59
2	609880	4.72	960618	.94	649263	5.66	350737	58
3	610164	4.72	960561	.94	649602	5.66	350398	57
4	610447	4.71	960505	.94	649942	5.65	350058	56
5	610729	4.71	960448	.94	650281	5.65	349719	55
6	611012	4.70	960392	.94	650620	5.65	349380	54
7	611294	4.70	960335	.94	650959	5.64	349041	53
8	611576	4.70	960279	.94	651297	5.64	348703	52
9	611858	4.69	960222	.94	651636	5.64	348364	51
10	612140	4.69	960165	.94	651974	5.63	348026	50
11	9.612421	4.69	9.960109	.95	9.652312	5.63	10.347688	49
12	612702	4.68	960052	.95	652650	5.63	347350	48
13	612983	4.68	959995	.95	652988	5.63	347012	47
14	613264	4.67	959938	.95	653326	5.62	346674	46
15	613545	4.67	959882	.95	653663	5.62	346337	45
16	613825	4.67	959825	.95	654000	5.62	346000	44
17	614105	4.66	959768	.95	654337	5.61	345663	43
18	614385	4.66	959711	.95	654674	5.61	345326	42
19	614665	4.66	959654	.95	655011	5.61	344989	41
20	614944	4.65	959596	.95	655348	5.61	344652	40
21	9.615223	4.65	9.959539	.95	9.655684	5.60	10.344316	39
22	615502	4.65	959482	.95	656020	5.60	343980	38
23	615781	4.64	959425	.95	656356	5.60	343644	37
24	616060	4.64	959368	.95	656692	5.59	343308	36
25	616338	4.64	959310	.96	657028	5.59	342972	35
26	616616	4.63	959253	.96	657364	5.59	342636	34
27	616894	4.63	959195	.96	657699	5.59	342301	33
28	617172	4.62	959138	.96	658034	5.58	341966	32
29	617450	4.62	959081	.96	658369	5.58	341631	31
30	617727	4.62	959023	.96	658704	5.58	341296	30
31	9.618004	4.61	9.958965	.96	9.659039	5.58	10.340961	29
32	618281	4.61	958908	.96	659373	5.57	340627	28
33	618558	4.61	958850	.96	659708	5.57	340292	27
34	618834	4.60	958792	.96	660042	5.57	339958	26
35	619110	4.60	958734	.96	660376	5.57	339624	25
36	619386	4.60	958677	.96	660710	5.56	339290	24
37	619662	4.59	958619	.96	661043	5.56	338957	23
38	619938	4.59	958561	.96	661377	5.56	338623	22
39	620213	4.59	958503	.97	661710	5.55	338290	21
40	620488	4.58	958445	.97	662043	5.55	337957	20
41	9.620763	4.58	9.958387	.97	9.662376	5.55	10.337624	19
42	621038	4.57	958329	.97	662709	5.54	337291	18
43	621313	4.57	958271	.97	663042	5.54	336958	17
44	621587	4.57	958213	.97	663375	5.54	336625	16
45	621861	4.56	958154	.97	663707	5.54	336293	15
46	622135	4.56	958096	.97	664039	5.53	335961	14
47	622409	4.56	958038	.97	664371	5.53	335629	13
48	622682	4.55	957979	.97	664703	5.53	335297	12
49	622956	4.55	957921	.97	665035	5.53	334965	11
50	623229	4.55	957863	.97	665366	5.52	334634	10
51	9.623502	4.54	9.957804	.97	9.665697	5.52	10.334303	9
52	623774	4.54	957746	.98	666029	5.52	333971	8
53	624047	4.54	957687	.98	666360	5.51	333640	7
54	624319	4.53	957628	.98	666691	5.51	333309	6
55	624591	4.53	957570	.98	667021	5.51	332979	5
56	624863	4.53	957511	.98	667352	5.51	332648	4
57	625135	4.52	957452	.98	667682	5.50	332318	3
58	625406	4.52	957393	.98	668013	5.50	331987	2
59	625677	4.52	957335	.98	668343	5.50	331657	1
60	625948	4.51	957276	.98	668672	5.50	331328	0
	Cosine	D.	Sine	D.	Cotang.	D.	Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9.625948	4.51	9.957276	.98	9.668673	5.50	10.331327	60
1	626219	4.51	957217	.98	669002	5.49	330998	59
2	626490	4.51	957158	.98	669332	5.49	330668	58
3	626760	4.50	957099	.98	669661	5.49	330339	57
4	627030	4.50	957040	.98	669991	5.48	330009	56
5	627300	4.50	956981	.98	670320	5.48	329689	55
6	627570	4.49	956921	.99	670649	5.48	329351	54
7	627840	4.49	956862	.99	670977	5.48	329023	53
8	628109	4.49	956803	.99	671306	5.47	328694	52
9	628378	4.48	956744	.99	671634	5.47	328366	51
10	628647	4.48	956684	.99	671963	5.47	328037	50
11	9.528916	4.47	9.956625	.99	9.672291	5.47	10.327709	49
12	629185	4.47	956566	.99	672619	5.46	327381	48
13	629453	4.47	956506	.99	672947	5.46	327053	47
14	629721	4.46	956447	.99	673274	5.46	326726	46
15	629989	4.46	956387	.99	673602	5.46	326398	45
16	630257	4.46	956327	.99	673929	5.45	326071	44
17	630524	4.46	956268	.99	674257	5.45	325743	43
18	630792	4.45	956208	1.00	674584	5.45	325416	42
19	631059	4.45	956148	1.00	674910	5.44	325090	41
20	631326	4.45	956089	1.00	675237	5.44	324763	40
21	9.631593	4.44	9.956029	1.00	9.675564	5.44	10.324436	39
22	631859	4.44	955969	1.00	675890	5.44	324110	38
23	632125	4.44	955909	1.00	676216	5.43	323784	37
24	632392	4.43	955849	1.00	676543	5.43	323457	36
25	632658	4.43	955789	1.00	676869	5.43	323131	35
26	632923	4.43	955729	1.00	677194	5.43	322806	34
27	633189	4.42	955669	1.00	677520	5.42	322480	33
28	633454	4.42	955609	1.00	677846	5.42	322154	32
29	633719	4.42	955548	1.00	678171	5.42	321829	31
30	633984	4.41	955488	1.00	678496	5.42	321504	30
31	9.634249	4.41	9.955428	1.01	9.678821	5.41	10.321179	29
32	634514	4.40	955368	1.01	679146	5.41	320854	28
33	634778	4.40	955307	1.01	679471	5.41	320529	27
34	635042	4.40	955247	1.01	679795	5.41	320205	26
35	635306	4.39	955186	1.01	680120	5.40	319880	25
36	635570	4.39	955126	1.01	680444	5.40	319556	24
37	635834	4.39	955065	1.01	680768	5.40	319232	23
38	636097	4.38	955005	1.01	681092	5.40	318908	22
39	636360	4.38	954944	1.01	681416	5.39	318584	21
40	636623	4.38	954883	1.01	681740	5.39	318260	20
41	9.636886	4.37	9.954823	1.01	9.682063	5.39	10.317937	19
42	637148	4.37	954762	1.01	682387	5.39	317613	18
43	637411	4.37	954701	1.01	682710	5.38	317290	17
44	637673	4.37	954640	1.01	683033	5.38	316967	16
45	637935	4.36	954579	1.01	683356	5.38	316644	15
46	638197	4.36	954518	1.02	683679	5.38	316321	14
47	638458	4.36	954457	1.02	684001	5.37	315999	13
48	638720	4.35	954396	1.02	684324	5.37	315676	12
49	638981	4.35	954335	1.02	684646	5.37	315354	11
50	639242	4.35	954274	1.02	684968	5.37	315032	10
51	9.639503	4.34	9.954213	1.02	9.685290	5.36	10.314710	9
52	639764	4.34	954152	1.02	685612	5.36	314388	8
53	640024	4.34	954090	1.02	685934	5.36	314066	7
54	640284	4.33	954029	1.02	686255	5.36	313745	6
55	640544	4.33	953968	1.02	686577	5.35	313423	5
56	640804	4.33	953906	1.02	686898	5.35	313102	4
57	641064	4.32	953845	1.02	687219	5.35	312781	3
58	641324	4.32	953783	1.02	687540	5.35	312460	2
59	641584	4.32	953722	1.03	687861	5.34	312139	1
60	641842	4.31	953660	1.03	688182	5.34	311818	0
	Cosine	D.	Sine	D.	Cotang.	D.	Tang.	M.



M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9.641842	4.31	9.953660	1.03	9.688182	5.34	10.311818	60
1	642101	4.31	953599	1.03	688502	5.34	311498	59
2	642360	4.31	953537	1.03	688823	5.34	311177	58
3	642618	4.30	953475	1.03	689143	5.33	310857	57
4	642877	4.30	953413	1.03	689463	5.33	310537	56
5	643135	4.30	953352	1.03	689783	5.33	310217	55
6	643393	4.30	953290	1.03	690103	5.33	309897	54
7	643650	4.29	953228	1.03	690423	5.33	309577	53
8	643908	4.29	953166	1.03	690742	5.32	309258	52
9	644165	4.29	953104	1.03	691062	5.32	308938	51
10	644423	4.28	953042	1.03	691381	5.32	308619	50
11	9.644680	4.28	9.952980	1.04	9.691700	5.31	10.308300	49
12	644936	4.28	952918	1.04	692019	5.31	307981	48
13	645193	4.27	952855	1.04	692338	5.31	307662	47
14	645450	4.27	952793	1.04	692656	5.31	307344	46
15	645706	4.27	952731	1.04	692975	5.31	307025	45
16	645962	4.26	952669	1.04	693293	5.30	306707	44
17	646218	4.26	952606	1.04	693612	5.30	306388	43
18	646474	4.26	952544	1.04	693930	5.30	306070	42
19	646729	4.25	952481	1.04	694248	5.30	305752	41
20	646984	4.25	952419	1.04	694566	5.29	305434	40
21	9.647240	4.25	9.952356	1.04	9.694883	5.29	10.305117	39
22	647494	4.24	952294	1.04	695201	5.29	304799	38
23	647749	4.24	952231	1.04	695518	5.29	304482	37
24	648004	4.24	952168	1.05	695836	5.29	304164	36
25	648258	4.24	952106	1.05	696153	5.28	303847	35
26	648512	4.23	952043	1.05	696470	5.28	303530	34
27	648766	4.23	951980	1.05	696787	5.28	303213	33
28	649020	4.23	951917	1.05	697103	5.28	302897	32
29	649274	4.22	951854	1.05	697420	5.27	302580	31
30	649527	4.22	951791	1.05	697736	5.27	302264	30
31	9.649781	4.22	9.951728	1.05	9.698053	5.27	10.301947	29
32	650034	4.22	951665	1.05	698369	5.27	301631	28
33	650287	4.21	951602	1.05	698685	5.26	301315	27
34	650539	4.21	951539	1.05	699001	5.26	300999	26
35	650792	4.21	951476	1.05	699316	5.26	300684	25
36	651044	4.20	951412	1.05	699632	5.26	300368	24
37	651297	4.20	951349	1.06	699947	5.26	300053	23
38	651549	4.20	951286	1.06	700263	5.25	299737	22
39	651800	4.19	951222	1.06	700578	5.25	299422	21
40	652052	4.19	951159	1.06	700893	5.25	299107	20
41	9.652304	4.19	9.951096	1.06	9.701208	5.24	10.298792	19
42	652555	4.18	951032	1.06	701523	5.24	298477	18
43	652806	4.18	950968	1.06	701837	5.24	298163	17
44	653057	4.18	950905	1.06	702152	5.24	297848	16
45	653308	4.18	950841	1.06	702466	5.24	297534	15
46	653558	4.17	950778	1.06	702780	5.23	297220	14
47	653808	4.17	950714	1.06	703095	5.23	296905	13
48	654059	4.17	950650	1.06	703409	5.23	296591	12
49	654309	4.16	950586	1.06	703723	5.23	296277	11
50	654558	4.16	950522	1.07	704036	5.22	295964	10
51	9.654808	4.16	9.950458	1.07	9.704350	5.22	10.295650	9
52	655058	4.16	950394	1.07	704663	5.22	295337	8
53	655307	4.15	950330	1.07	704977	5.22	295023	7
54	655556	4.15	950266	1.07	705290	5.22	294710	6
55	655805	4.15	950202	1.07	705603	5.21	294397	5
56	656054	4.14	950138	1.07	705916	5.21	294084	4
57	656302	4.14	950074	1.07	706228	5.21	293772	3
58	656551	4.14	950010	1.07	706541	5.21	293459	2
59	656799	4.13	949945	1.07	706854	5.21	293146	1
60	657047	4.13	949881	1.07	707166	5.20	292834	0
	Cosine	D.	Sine	D.	Cotang.	D.	Tang.	M.



M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9.657047	4.13	9.949881	1.07	9.707166	5.20	10.292834	60
1	657295	4.13	949816	1.07	707478	5.20	292522	59
2	657542	4.12	949752	1.07	707790	5.20	292210	58
3	657790	4.12	949688	1.08	708102	5.20	291898	57
4	658037	4.12	949623	1.08	708414	5.19	291586	56
5	658284	4.12	949558	1.08	708726	5.19	291274	55
6	658531	4.11	949494	1.08	709037	5.19	290963	54
7	658778	4.11	949429	1.08	709349	5.19	290651	53
8	659025	4.11	949364	1.08	709660	5.19	290340	52
9	659271	4.10	949300	1.08	709971	5.18	290029	51
10	659517	4.10	949235	1.08	710282	5.18	289718	50
11	9.659763	4.10	9.949170	1.08	9.710593	5.18	10.289407	49
12	660009	4.09	949105	1.08	710904	5.18	289096	48
13	660255	4.09	949040	1.08	711215	5.18	288785	47
14	660501	4.09	948975	1.08	711525	5.17	288475	46
15	660746	4.09	948910	1.08	711836	5.17	288164	45
16	660991	4.08	948845	1.08	712146	5.17	287854	44
17	661236	4.08	948780	1.09	712456	5.17	287544	43
18	661481	4.08	948715	1.09	712766	5.16	287234	42
19	661726	4.07	948650	1.09	713076	5.16	286924	41
20	661970	4.07	948584	1.09	713386	5.16	286614	40
21	9.662214	4.07	9.948519	1.09	9.713696	5.16	10.286304	39
22	662459	4.07	948454	1.09	714005	5.16	285995	38
23	662703	4.06	948388	1.09	714314	5.15	285686	37
24	662946	4.06	948323	1.09	714624	5.15	285376	36
25	663190	4.06	948257	1.09	714933	5.15	285067	35
26	663433	4.05	948192	1.09	715242	5.15	284758	34
27	663677	4.05	948126	1.09	715551	5.14	284449	33
28	663920	4.05	948060	1.09	715860	5.14	284140	32
29	664163	4.05	947995	1.10	716168	5.14	283832	31
30	664406	4.04	947929	1.10	716477	5.14	283523	30
31	9.664648	4.04	9.947863	1.10	9.716785	5.14	10.283215	29
32	664891	4.04	947797	1.10	717093	5.13	282907	28
33	665133	4.03	947731	1.10	717401	5.13	282599	27
34	665375	4.03	947665	1.10	717709	5.13	282291	26
35	665617	4.03	947600	1.10	718017	5.13	281983	25
36	665859	4.02	947533	1.10	718325	5.13	281670	24
37	666100	4.02	947467	1.10	718633	5.12	281367	23
38	666342	4.02	947401	1.10	718940	5.12	281060	22
39	666583	4.02	947335	1.10	719248	5.12	280752	21
40	666824	4.01	947269	1.10	719555	5.12	280445	20
41	9.667065	4.01	9.947203	1.10	9.719862	5.12	10.280138	19
42	667305	4.01	947136	1.11	720169	5.11	279831	18
43	667546	4.01	947070	1.11	720476	5.11	279524	17
44	667786	4.00	947004	1.11	720783	5.11	279217	16
45	668027	4.00	946937	1.11	721089	5.11	278911	15
46	668267	4.00	946871	1.11	721396	5.11	278604	14
47	668506	3.99	946804	1.11	721702	5.10	278298	13
48	668746	3.99	946738	1.11	722009	5.10	277991	12
49	668986	3.99	946671	1.11	722315	5.10	277685	11
50	669225	3.99	946604	1.11	722621	5.10	277379	10
51	9.669464	3.98	9.946538	1.11	9.722927	5.10	10.277073	9
52	669703	3.98	946471	1.11	723232	5.09	276768	8
53	669942	3.98	946404	1.11	723538	5.09	276462	7
54	670181	3.97	946337	1.11	723844	5.09	276156	6
55	670419	3.97	946270	1.12	724149	5.09	275851	5
56	670658	3.97	946203	1.12	724454	5.09	275546	4
57	670896	3.97	946136	1.12	724759	5.08	275241	3
58	671134	3.96	946069	1.12	725065	5.08	274935	2
59	671372	3.96	946002	1.12	725369	5.08	274631	1
60	671609	3.96	945935	1.12	725674	5.08	274326	0
	Cosine	D.	Sine	D.	Cotang.	D.	Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9.671609	3.96	9.945935	1.12	9.725674	5.08	10.274326	60
1	671847	3.95	945868	1.12	725979	5.08	274021	59
2	672084	3.95	945800	1.12	726284	5.07	273716	58
3	672321	3.95	945733	1.12	726588	5.07	273412	57
4	672558	3.95	945666	1.12	726892	5.07	273108	56
5	672795	3.94	945598	1.12	727197	5.07	272803	55
6	673032	3.94	945531	1.12	727501	5.07	272499	54
7	673268	3.94	945464	1.13	727805	5.06	272195	53
8	673505	3.94	945396	1.13	728109	5.06	271891	52
9	673741	3.93	945328	1.13	728412	5.06	271588	51
10	673977	3.93	945261	1.13	728716	5.06	271284	50
11	9.674213	3.93	9.945193	1.13	9.729020	5.06	10.270280	49
12	674448	3.92	945125	1.13	729323	5.05	270677	48
13	674684	3.92	945058	1.13	729626	5.05	270374	47
14	674919	3.92	944990	1.13	729929	5.05	270071	46
15	675155	3.92	944922	1.13	730233	5.05	269767	45
16	675390	3.91	944854	1.13	730535	5.05	269465	44
17	675624	3.91	944786	1.13	730838	5.04	269162	43
18	675859	3.91	944718	1.13	731141	5.04	268859	42
19	676094	3.91	944650	1.13	731444	5.04	268556	41
20	676328	3.90	944582	1.14	731746	5.04	268254	40
21	9.676562	3.90	9.944514	1.14	9.732048	5.04	10.267952	39
22	676796	3.90	944446	1.14	732351	5.03	267649	38
23	677030	3.90	944377	1.14	732653	5.03	267347	37
24	677264	3.89	944309	1.14	732955	5.03	267045	36
25	677498	3.89	944241	1.14	733257	5.03	266743	35
26	677731	3.89	944172	1.14	733558	5.03	266442	34
27	677964	3.88	944104	1.14	733860	5.02	266140	33
28	678197	3.88	944036	1.14	734162	5.02	265838	32
29	678430	3.88	943967	1.14	734463	5.02	265537	31
30	678663	3.88	943899	1.14	734764	5.02	265236	30
31	9.678895	3.87	9.943830	1.14	9.735066	5.02	10.264934	29
32	679128	3.87	943761	1.14	735367	5.02	264633	28
33	679360	3.87	943693	1.15	735668	5.01	264332	27
34	679592	3.87	943624	1.15	735969	5.01	264031	26
35	679824	3.86	943555	1.15	736269	5.01	263731	25
36	680056	3.86	943486	1.15	736570	5.01	263430	24
37	680288	3.86	943417	1.15	736871	5.01	263129	23
38	680519	3.85	943348	1.15	737171	5.00	262829	22
39	680750	3.85	943279	1.15	737471	5.00	262529	21
40	680982	3.85	943210	1.15	737771	5.00	262229	20
41	9.681213	3.85	9.943141	1.15	9.738071	5.00	10.261929	19
42	681443	3.84	943072	1.15	738371	5.00	261629	18
43	681674	3.84	943003	1.15	738671	4.99	261329	17
44	681905	3.84	942934	1.15	738971	4.99	261029	16
45	682135	3.84	942864	1.15	739271	4.99	260729	15
46	682365	3.83	942795	1.16	739570	4.99	260430	14
47	682595	3.83	942726	1.16	739870	4.99	260130	13
48	682825	3.83	942656	1.16	740169	4.99	259831	12
49	683055	3.83	942587	1.16	740468	4.98	259532	11
50	683284	3.82	942517	1.16	740767	4.98	259233	10
51	9.683514	3.82	9.942448	1.16	9.741066	4.98	10.258934	9
52	683743	3.82	942378	1.16	741365	4.98	258635	8
53	683972	3.82	942308	1.16	741664	4.98	258336	7
54	684201	3.81	942239	1.16	741962	4.97	258038	6
55	684430	3.81	942169	1.16	742261	4.97	257739	5
56	684658	3.81	942099	1.16	742559	4.97	257441	4
57	684887	3.80	942029	1.16	742858	4.97	257142	3
58	685115	3.80	941959	1.16	743156	4.97	256844	2
59	685343	3.80	941889	1.17	743454	4.97	256546	1
60	685571	3.80	941819	1.17	743752	4.96	256248	0
	Cosine	D.	Sine	D.	Cotang.	D.	Tang.	M.



M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9.685571	3.80	9.941819	1.17	9.743752	4.96	10.256248	60
1	685799	3.79	941749	1.17	744050	4.96	255950	59
2	686027	3.79	941679	1.17	744348	4.96	255652	58
3	686254	3.79	941609	1.17	744645	4.96	255355	57
4	686482	3.79	941539	1.17	744943	4.96	255057	56
5	686709	3.78	941469	1.17	745240	4.96	254760	55
6	686936	3.78	941398	1.17	745538	4.95	254462	54
7	687163	3.78	941328	1.17	745835	4.95	254165	53
8	687389	3.78	941258	1.17	746132	4.95	253868	52
9	687616	3.77	941187	1.17	746429	4.95	253571	51
10	687843	3.77	941117	1.17	746726	4.95	253274	50
11	9.688069	3.77	9.941046	1.18	9.747023	4.94	10.252977	49
12	688295	3.77	940975	1.18	747319	4.94	252681	48
13	688521	3.76	940905	1.18	747616	4.94	252384	47
14	688747	3.76	940834	1.18	747913	4.94	252087	46
15	688972	3.76	940763	1.18	748209	4.94	251791	45
16	689198	3.76	940693	1.18	748505	4.93	251495	44
17	689423	3.75	940622	1.18	748801	4.93	251199	43
18	689648	3.75	940551	1.18	749097	4.93	250903	42
19	689873	3.75	940480	1.18	749393	4.93	250607	41
20	690098	3.75	940409	1.18	749689	4.93	250311	40
21	9.690323	3.74	9.940338	1.18	9.749985	4.93	10.250015	39
22	690548	3.74	940267	1.18	750281	4.92	249719	38
23	690772	3.74	940196	1.18	750576	4.92	249424	37
24	690996	3.74	940125	1.19	750872	4.92	249128	36
25	691220	3.73	940054	1.19	751167	4.92	248833	35
26	691444	3.73	939982	1.19	751462	4.92	248538	34
27	691668	3.73	939911	1.19	751757	4.92	248243	33
28	691892	3.73	939840	1.19	752052	4.91	247948	32
29	692115	3.72	939768	1.19	752347	4.91	247653	31
30	692339	3.72	939697	1.19	752642	4.91	247358	30
31	9.692562	3.72	9.939625	1.19	9.752937	4.91	10.247063	29
32	692785	3.71	939554	1.19	753231	4.91	246769	28
33	693008	3.71	939482	1.19	753526	4.91	246474	27
34	693231	3.71	939410	1.19	753820	4.90	246180	26
35	693453	3.71	939339	1.19	754115	4.90	245885	25
36	693676	3.70	939267	1.20	754409	4.90	245591	24
37	693898	3.70	939195	1.20	754703	4.90	245297	23
38	694120	3.70	939123	1.20	754997	4.90	245003	22
39	694342	3.70	939052	1.20	755291	4.90	244709	21
40	694564	3.69	938980	1.20	755585	4.89	244415	20
41	9.694786	3.69	9.938908	1.20	9.755878	4.89	10.244122	19
42	695007	3.69	938836	1.20	756172	4.89	243828	18
43	695229	3.69	938763	1.20	756465	4.89	243535	17
44	695450	3.68	938691	1.20	756759	4.89	243241	16
45	695671	3.68	938619	1.20	757052	4.89	242948	15
46	695892	3.68	938547	1.20	757345	4.88	242655	14
47	696113	3.68	938475	1.20	757638	4.88	242362	13
48	696334	3.67	938402	1.21	757931	4.88	242069	12
49	696554	3.67	938330	1.21	758224	4.88	241776	11
50	696775	3.67	938258	1.21	758517	4.88	241483	10
51	9.696995	3.67	9.938185	1.21	9.758810	4.88	10.241190	9
52	697215	3.66	938113	1.21	759102	4.87	240898	8
53	697435	3.65	938040	1.21	759395	4.87	240605	7
54	697654	3.66	937967	1.21	759687	4.87	240313	6
55	697874	3.66	937895	1.21	759979	4.87	240021	5
56	698094	3.65	937822	1.21	760272	4.87	239728	4
57	698313	3.65	937749	1.21	760564	4.87	239436	3
58	698532	3.65	937676	1.21	760856	4.86	239144	2
59	698751	3.65	937604	1.21	761148	4.86	238852	1
60	698970	3.64	937531	1.21	761439	4.86	238561	0
	Cosine	D.	Sine	D.	Cotang.	D.	Tang.	M.



M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9.698970	3.64	9.937531	1.21	9.761439	4.86	10.238561	60
1	699189	3.64	937458	1.22	761731	4.86	238269	59
2	699407	3.64	937385	1.22	762023	4.86	237977	58
3	699626	3.64	937312	1.22	762314	4.86	237686	57
4	699844	3.63	937238	1.22	762606	4.85	237394	56
5	700062	3.63	937165	1.22	762897	4.85	237103	55
6	700280	3.63	937092	1.22	763188	4.85	236812	54
7	700498	3.63	937019	1.22	763479	4.85	236521	53
8	700716	3.63	936946	1.22	763770	4.85	236230	52
9	700933	3.62	936872	1.22	764061	4.85	235939	51
10	701151	3.62	936799	1.22	764352	4.84	235648	50
11	9.701368	3.62	9.936725	1.22	9.764643	4.84	10.235357	49
12	701585	3.62	936652	1.23	764933	4.84	235067	48
13	701802	3.61	936578	1.23	765224	4.84	234776	47
14	702019	3.61	936505	1.23	765514	4.84	234486	46
15	702236	3.61	936431	1.23	765805	4.84	234195	45
16	702452	3.61	936357	1.23	766095	4.84	233905	44
17	702669	3.60	936284	1.23	766385	4.83	233615	43
18	702885	3.60	936210	1.23	766675	4.83	233325	42
19	703101	3.60	936136	1.23	766965	4.83	233035	41
20	703317	3.60	936062	1.23	767255	4.83	232745	40
21	9.703533	3.59	9.935988	1.23	9.767545	4.83	10.232455	39
22	703749	3.59	935914	1.23	767834	4.83	232166	38
23	703964	3.59	935840	1.23	768124	4.82	231876	37
24	704179	3.59	935766	1.24	768413	4.82	231587	36
25	704395	3.59	935692	1.24	768703	4.82	231297	35
26	704610	3.58	935618	1.24	768992	4.82	231008	34
27	704825	3.58	935543	1.24	769281	4.82	230719	33
28	705040	3.58	935469	1.24	769570	4.82	230430	32
29	705254	3.58	935395	1.24	769860	4.81	230140	31
30	705469	3.57	935320	1.24	770148	4.81	229852	30
31	9.705683	3.57	9.935246	1.24	9.770437	4.81	10.229563	29
32	705898	3.57	935171	1.24	770726	4.81	229274	28
33	706112	3.57	935097	1.24	771015	4.81	228985	27
34	706326	3.56	935022	1.24	771303	4.81	228697	26
35	706539	3.56	934948	1.24	771592	4.81	228408	25
36	706753	3.56	934873	1.24	771880	4.80	228120	24
37	706967	3.56	934798	1.25	772168	4.80	227832	23
38	707180	3.55	934723	1.25	772457	4.80	227543	22
39	707393	3.55	934649	1.25	772745	4.80	227255	21
40	707606	3.55	934574	1.25	773033	4.80	226967	20
41	9.707819	3.55	9.934499	1.25	9.773321	4.80	10.226679	19
42	708032	3.54	934424	1.25	773608	4.79	226392	18
43	708245	3.54	934349	1.25	773896	4.79	226104	17
44	708458	3.54	934274	1.25	774184	4.79	225816	16
45	708670	3.54	934199	1.25	774471	4.79	225529	15
46	708882	3.53	934123	1.25	774759	4.79	225241	14
47	709094	3.53	934048	1.25	775046	4.79	224954	13
48	709306	3.53	933973	1.25	775333	4.78	224667	12
49	709518	3.53	933898	1.26	775621	4.78	224379	11
50	709730	3.53	933822	1.26	775908	4.78	224092	10
51	9.709941	3.52	9.933747	1.26	9.776195	4.78	10.223805	9
52	710153	3.52	933671	1.26	776482	4.78	223518	8
53	710364	3.52	933596	1.26	776769	4.78	223231	7
54	710575	3.52	933520	1.26	777055	4.78	222945	6
55	710786	3.51	933445	1.26	777342	4.78	222658	5
56	710997	3.51	933369	1.26	777628	4.77	222372	4
57	711208	3.51	933293	1.26	777915	4.77	222085	3
58	711419	3.51	933217	1.26	778201	4.77	221799	2
59	711629	3.50	933141	1.26	778487	4.77	221512	1
60	711839	3.50	933066	1.26	778774	4.77	221226	0
	Cosine	D.	Sine	D.	Cotang.	D.	Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9.711839	3.50	9.933066	1.26	9.778774	4.77	10.221226	60
1	712050	3.50	932990	1.27	779060	4.77	220940	59
2	712260	3.50	932914	1.27	779346	4.76	220654	58
3	712469	3.49	932838	1.27	779632	4.76	220368	57
4	712679	3.49	932762	1.27	779918	4.76	220082	56
5	712889	3.49	932685	1.27	780203	4.76	219797	55
6	713098	3.49	932609	1.27	780489	4.76	219511	54
7	713308	3.49	932533	1.27	780775	4.76	219225	53
8	713517	3.48	932457	1.27	781060	4.76	218940	52
9	713726	3.48	932380	1.27	781346	4.75	218654	51
10	713935	3.48	932304	1.27	781631	4.75	218369	50
11	9.714144	3.48	9.932228	1.27	9.781916	4.75	10.218084	49
12	714352	3.47	932151	1.27	782201	4.75	217799	48
13	714561	3.47	932075	1.28	782486	4.75	217514	47
14	714769	3.47	931998	1.28	782771	4.75	217229	46
15	714978	3.47	931921	1.28	783056	4.75	216944	45
16	715186	3.47	931845	1.28	783341	4.75	216659	44
17	715394	3.46	931768	1.28	783626	4.74	216374	43
18	715602	3.46	931691	1.28	783910	4.74	216090	42
19	715809	3.46	931614	1.28	784195	4.74	215805	41
20	716017	3.46	931537	1.28	784479	4.74	215521	40
21	9.716224	3.45	9.931460	1.28	9.784764	4.74	10.215236	39
22	716432	3.45	931383	1.28	785048	4.74	214952	38
23	716639	3.45	931306	1.28	785332	4.73	214668	37
24	716846	3.45	931229	1.29	785616	4.73	214384	36
25	717053	3.45	931152	1.29	785900	4.73	214100	35
26	717259	3.44	931075	1.29	786184	4.73	213816	34
27	717466	3.44	930998	1.29	786468	4.73	213532	33
28	717673	3.44	930921	1.29	786752	4.73	213248	32
29	717879	3.44	930843	1.29	787036	4.73	212964	31
30	718085	3.43	930766	1.29	787319	4.72	212681	30
31	9.718291	3.43	9.930688	1.29	9.787603	4.72	10.212397	29
32	718497	3.43	930611	1.29	787886	4.72	212114	28
33	718703	3.43	930533	1.29	788170	4.72	211830	27
34	718909	3.43	930456	1.29	788453	4.72	211547	26
35	719114	3.42	930378	1.29	788736	4.72	211264	25
36	719320	3.42	930300	1.30	789019	4.72	210981	24
37	719525	3.42	930223	1.30	789302	4.71	210698	23
38	719730	3.42	930145	1.30	789585	4.71	210415	22
39	719935	3.41	930067	1.30	789868	4.71	210132	21
40	720140	3.41	929989	1.30	790151	4.71	209849	20
41	9.720345	3.41	9.929911	1.30	9.790433	4.71	10.209567	19
42	720549	3.41	929833	1.30	790716	4.71	209284	18
43	720754	3.40	929755	1.30	790999	4.71	209001	17
44	720958	3.40	929677	1.30	791281	4.71	208719	16
45	721162	3.40	929599	1.30	791563	4.70	208437	15
46	721366	3.40	929521	1.30	791846	4.70	208154	14
47	721570	3.40	929442	1.30	792128	4.70	207872	13
48	721774	3.39	929364	1.31	792410	4.70	207590	12
49	721978	3.39	929286	1.31	792692	4.70	207308	11
50	722181	3.39	929207	1.31	792974	4.70	207026	10
51	9.722385	3.39	9.929129	1.31	9.793256	4.70	10.206744	9
52	722588	3.39	929050	1.31	793538	4.69	206462	8
53	722791	3.38	928972	1.31	793819	4.69	206181	7
54	722994	3.38	928893	1.31	794101	4.69	205899	6
55	723197	3.38	928815	1.31	794383	4.69	205617	5
56	723400	3.38	928736	1.31	794664	4.69	205336	4
57	723603	3.37	928657	1.31	794945	4.69	205055	3
58	723805	3.37	928578	1.31	795227	4.69	204773	2
59	724007	3.37	928499	1.31	795508	4.68	204492	1
60	724210	3.37	928420	1.31	795789	4.68	204211	0
	Cosine	D.	Sine	D.	Cotang.	D.	Tang.	M.



M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	724210	3.37	9.928420	1.32	9.795789	4.68	10.204211	60
1	724412	3.37	928342	1.32	796070	4.68	203930	59
2	724614	3.36	928263	1.32	796351	4.68	203649	58
3	724816	3.36	928183	1.32	796632	4.68	203368	57
4	725017	3.36	928104	1.32	796913	4.68	203087	56
5	725219	3.36	928025	1.32	797194	4.68	202806	55
6	725420	3.35	927946	1.32	797475	4.68	202525	54
7	725622	3.35	927867	1.32	797755	4.68	202245	53
8	725823	3.35	927787	1.32	798036	4.67	201964	52
9	726024	3.35	927708	1.32	798316	4.67	201684	51
10	726225	3.35	927629	1.32	798596	4.67	201404	50
11	9.726426	3.34	9.927549	1.32	9.798877	4.67	10.201123	49
12	726626	3.34	927470	1.33	799157	4.67	200843	48
13	726827	3.34	927390	1.33	799437	4.67	200563	47
14	727027	3.34	927310	1.33	799717	4.67	200283	46
15	727228	3.34	927231	1.33	799997	4.66	200003	45
16	727428	3.33	927151	1.33	800277	4.66	199723	44
17	727628	3.33	927071	1.33	800557	4.66	199443	43
18	727828	3.33	926991	1.33	800836	4.66	199164	42
19	728027	3.33	926911	1.33	801116	4.66	198884	41
20	728227	3.33	926831	1.33	801396	4.66	198604	40
21	728427	3.32	9.926751	1.33	9.801675	4.66	10.198325	39
22	728626	3.32	926671	1.33	801955	4.66	198045	38
23	728825	3.32	926591	1.33	802234	4.65	197766	37
24	729024	3.32	926511	1.34	802513	4.65	197487	36
25	729223	3.31	926431	1.34	802792	4.65	197208	35
26	729422	3.31	926351	1.34	803072	4.65	196928	34
27	729621	3.31	926270	1.34	803351	4.65	196649	33
28	729820	3.31	926190	1.34	803630	4.65	196370	32
29	730018	3.30	926110	1.34	803908	4.65	196092	31
30	730216	3.30	926029	1.34	804187	4.65	195813	30
31	730415	3.30	9.925949	1.34	9.804466	4.64	10.195534	29
32	730613	3.30	925868	1.34	804745	4.64	195255	28
33	730811	3.30	925788	1.34	805023	4.64	194977	27
34	731009	3.29	925707	1.34	805302	4.64	194698	26
35	731206	3.29	925626	1.34	805580	4.64	194420	25
36	731404	3.29	925545	1.35	805859	4.64	194141	24
37	731602	3.29	925465	1.35	806137	4.64	193863	23
38	731799	3.29	925384	1.35	806415	4.63	193585	22
39	731996	3.28	925303	1.35	806693	4.63	193307	21
40	732193	3.28	925222	1.35	806971	4.63	193029	20
41	9.732390	3.28	9.925141	1.35	9.807249	4.63	10.192751	19
42	732587	3.28	925060	1.35	807527	4.63	192473	18
43	732784	3.28	924979	1.35	807805	4.63	192195	17
44	732980	3.27	924897	1.35	808083	4.63	191917	16
45	733177	3.27	924816	1.35	808361	4.63	191639	15
46	733373	3.27	924735	1.36	808638	4.62	191362	14
47	733569	3.27	924654	1.36	808916	4.62	191084	13
48	733765	3.27	924572	1.36	809193	4.62	190807	12
49	733961	3.26	924491	1.36	809471	4.62	190529	11
50	734157	3.26	924409	1.36	809748	4.62	190252	10
51	9.734353	3.26	9.924328	1.36	9.810025	4.62	10.189975	9
52	734549	3.26	924246	1.36	810302	4.62	189698	8
53	734744	3.25	924164	1.36	810580	4.62	189420	7
54	734939	3.25	924083	1.36	810857	4.62	189143	6
55	735135	3.25	924001	1.36	811134	4.61	188866	5
56	735330	3.25	923919	1.36	811410	4.61	188590	4
57	735525	3.25	923837	1.36	811687	4.61	188313	3
58	735719	3.24	923755	1.37	811964	4.61	188036	2
59	735914	3.24	923673	1.37	812241	4.61	187759	1
60	736109	3.24	923591	1.37	812517	4.61	187483	0
	Cosine	D.	Sine	D.	Cotang.	D.	Tang.	M.



M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9.736109	3.24	9.923591	1.37	9.812517	4.61	10.187482	60
1	736303	3.24	923509	1.37	812794	4.61	187206	59
2	736498	3.24	923427	1.37	813070	4.61	186930	58
3	736692	3.23	923345	1.37	813347	4.60	186653	57
4	736886	3.23	923263	1.37	813623	4.60	186377	56
5	737080	3.23	923181	1.37	813899	4.60	186101	55
6	737274	3.23	923098	1.37	814175	4.60	185825	54
7	737467	3.23	923016	1.37	814452	4.60	185548	53
8	737661	3.22	922933	1.37	814728	4.60	185272	52
9	737855	3.22	922851	1.37	815004	4.60	184996	51
10	738048	3.22	922768	1.38	815279	4.60	184721	50
11	9.738241	3.22	9.922686	1.38	9.815555	4.59	10.184445	49
12	738434	3.22	922603	1.38	815831	4.59	184169	48
13	738627	3.21	922520	1.38	816107	4.59	183893	47
14	738820	3.21	922438	1.38	816382	4.59	183618	46
15	739013	3.21	922355	1.38	816658	4.59	183342	45
16	739206	3.21	922272	1.38	816933	4.59	183067	44
17	739398	3.21	922189	1.38	817209	4.59	182791	43
18	739590	3.20	922106	1.38	817484	4.59	182516	42
19	739783	3.20	922023	1.38	817759	4.59	182241	41
20	739975	3.20	921940	1.38	818035	4.58	181965	40
21	9.740167	3.20	9.921857	1.39	9.818310	4.58	10.181690	39
22	740359	3.20	921774	1.39	818585	4.58	181415	38
23	740550	3.19	921691	1.39	818860	4.58	181140	37
24	740742	3.19	921607	1.39	819135	4.58	180865	36
25	740934	3.19	921524	1.39	819410	4.58	180590	35
26	741125	3.19	921441	1.39	819684	4.58	180316	34
27	741316	3.19	921357	1.39	819959	4.58	180041	33
28	741508	3.18	921274	1.39	820234	4.58	179766	32
29	741699	3.18	921190	1.39	820508	4.57	179492	31
30	741889	3.18	921107	1.39	820783	4.57	179217	30
31	9.742080	3.18	9.921023	1.39	9.821057	4.57	10.178943	29
32	742271	3.18	920939	1.40	821332	4.57	178668	28
33	742462	3.17	920856	1.40	821606	4.57	178394	27
34	742652	3.17	920772	1.40	821880	4.57	178120	26
35	742842	3.17	920688	1.40	822154	4.57	177846	25
36	743033	3.17	920604	1.40	822429	4.57	177571	24
37	743223	3.17	920520	1.40	822703	4.57	177297	23
38	743413	3.16	920436	1.40	822977	4.56	177023	22
39	743602	3.16	920352	1.40	823250	4.56	176750	21
40	743792	3.16	920268	1.40	823524	4.56	176476	20
41	9.743982	3.16	9.920184	1.40	9.823798	4.56	10.176202	19
42	744171	3.16	920099	1.40	824072	4.56	175928	18
43	744361	3.15	920015	1.40	824345	4.56	175655	17
44	744550	3.15	919931	1.41	824619	4.56	175381	16
45	744739	3.15	919846	1.41	824893	4.56	175107	15
46	744928	3.15	919762	1.41	825166	4.56	174834	14
47	745117	3.15	919677	1.41	825439	4.55	174561	13
48	745306	3.14	919593	1.41	825713	4.55	174287	12
49	745494	3.14	919508	1.41	825986	4.55	174014	11
50	745683	3.14	919424	1.41	826259	4.55	173741	10
51	9.745871	3.14	9.919339	1.41	9.826532	4.55	10.173468	9
52	746059	3.14	919254	1.41	826805	4.55	173195	8
53	746248	3.13	919169	1.41	827078	4.55	172922	7
54	746436	3.13	919085	1.41	827351	4.55	172649	6
55	746624	3.13	919000	1.41	827624	4.55	172376	5
56	746812	3.13	918915	1.42	827897	4.54	172103	4
57	746999	3.13	918830	1.42	828170	4.54	171830	3
58	747187	3.12	918745	1.42	828442	4.54	171558	2
59	747374	3.12	918659	1.42	828715	4.54	171285	1
60	747562	3.12	918574	1.42	828987	4.54	171013	0
	Cosine	D.	Sine	D.	Cotang.	D.	Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9.747562	3.12	9.918574	1.42	9.828987	4.54	10.171013	60
1	747749	3.12	918489	1.42	829260	4.54	170740	59
2	747936	3.12	918404	1.42	829532	4.54	170468	58
3	748123	3.11	918318	1.42	829805	4.54	170195	57
4	748310	3.11	918233	1.42	830077	4.54	169923	56
5	748497	3.11	918147	1.42	830349	4.53	169651	55
6	748683	3.11	918062	1.42	830621	4.53	169379	54
7	748870	3.11	917976	1.43	830893	4.53	169107	53
8	749056	3.10	917891	1.43	831165	4.53	168835	52
9	749243	3.10	917805	1.43	831437	4.53	168563	51
10	749429	3.10	917719	1.43	831709	4.53	168291	50
11	9.749615	3.10	9.917634	1.43	9.831981	4.53	10.168019	49
12	749801	3.10	917548	1.43	832253	4.53	167747	48
13	749987	3.09	917462	1.43	832525	4.53	167475	47
14	750172	3.09	917376	1.43	832796	4.53	167204	46
15	750358	3.09	917290	1.43	833068	4.52	166932	45
16	750543	3.09	917204	1.43	833339	4.52	166661	44
17	750729	3.09	917118	1.44	833611	4.52	166389	43
18	750914	3.08	917032	1.44	833882	4.52	166118	42
19	751099	3.08	916946	1.44	834154	4.52	165846	41
20	751284	3.08	916859	1.44	834425	4.52	165575	40
21	9.751469	3.08	9.916773	1.44	9.834696	4.52	10.165304	39
22	751654	3.08	916687	1.44	834967	4.52	165033	38
23	751839	3.08	916600	1.44	835238	4.52	164762	37
24	752023	3.07	916514	1.44	835509	4.52	164491	36
25	752208	3.07	916427	1.44	835780	4.51	164220	35
26	752392	3.07	916341	1.44	836051	4.51	163949	34
27	752576	3.07	916254	1.44	836322	4.51	163678	33
28	752760	3.07	916167	1.45	836593	4.51	163407	32
29	752944	3.06	916081	1.45	836864	4.51	163136	31
30	753128	3.06	915994	1.45	837134	4.51	162866	30
31	9.753312	3.06	9.915907	1.45	9.837405	4.51	10.162595	29
32	753495	3.06	915820	1.45	837675	4.51	162325	28
33	753679	3.06	915733	1.45	837946	4.51	162054	27
34	753862	3.05	915646	1.45	838216	4.51	161784	26
35	754046	3.05	915559	1.45	838487	4.50	161513	25
36	754229	3.05	915472	1.45	838757	4.50	161243	24
37	754412	3.05	915385	1.45	839027	4.50	160973	23
38	754595	3.05	915297	1.45	839297	4.50	160703	22
39	754778	3.04	915210	1.45	839568	4.50	160432	21
40	754960	3.04	915123	1.46	839838	4.50	160162	20
41	9.755143	3.04	9.915035	1.46	9.840108	4.50	10.159892	19
42	755326	3.04	914948	1.46	840378	4.50	159622	18
43	755508	3.04	914860	1.46	840647	4.50	159353	17
44	755690	3.04	914773	1.46	840917	4.49	159083	16
45	755872	3.03	914685	1.46	841187	4.49	158813	15
46	756054	3.03	914598	1.46	841457	4.49	158543	14
47	756236	3.03	914510	1.46	841726	4.49	158274	13
48	756418	3.03	914422	1.46	841996	4.49	158004	12
49	756600	3.03	914334	1.46	842266	4.49	157734	11
50	756782	3.02	914246	1.47	842535	4.49	157465	10
51	9.756963	3.02	9.914158	1.47	9.842805	4.49	10.157195	9
52	757144	3.02	914070	1.47	843074	4.49	156926	8
53	757326	3.02	913982	1.47	843343	4.49	156657	7
54	757507	3.02	913894	1.47	843612	4.49	156388	6
55	757688	3.01	913806	1.47	843882	4.48	156118	5
56	757869	3.01	913718	1.47	844151	4.48	155849	4
57	758050	3.01	913630	1.47	844420	4.48	155580	3
58	758230	3.01	913541	1.47	844689	4.48	155311	2
59	758411	3.01	913453	1.47	844958	4.48	155042	1
60	758591	3.01	913365	1.47	845227	4.48	154773	0
	Cosine	D.	Sine	D.	Cotang.	D.	Tang.	M.



M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9.758591	3.01	9.913365	1.47	9.845227	4.48	10.154773	60
1	758772	3.00	913276	1.47	845496	4.48	154504	59
2	758952	3.00	913187	1.48	845764	4.48	154236	58
3	759132	3.00	913099	1.48	846033	4.48	153967	57
4	759312	3.00	913010	1.48	846302	4.48	153698	56
5	759492	3.00	912922	1.48	846570	4.47	153430	55
6	759672	2.99	912833	1.48	846839	4.47	153161	54
7	759852	2.99	912744	1.48	847107	4.47	152893	53
8	760031	2.99	912655	1.48	847376	4.47	152624	52
9	760211	2.99	912566	1.48	847644	4.47	152356	51
10	760390	2.99	912477	1.48	847913	4.47	152087	50
11	9.760569	2.98	9.912388	1.48	9.848181	4.47	10.151819	49
12	760748	2.98	912299	1.49	848449	4.47	151551	48
13	760927	2.98	912210	1.49	848717	4.47	151283	47
14	761106	2.98	912121	1.49	848986	4.47	151014	46
15	761285	2.98	912031	1.49	849254	4.47	150746	45
16	761464	2.98	911942	1.49	849522	4.47	150478	44
17	761642	2.97	911853	1.49	849790	4.46	150210	43
18	761821	2.97	911763	1.49	850058	4.46	149942	42
19	761999	2.97	911674	1.49	850325	4.46	149675	41
20	762177	2.97	911584	1.49	850593	4.46	149407	40
21	9.762356	2.97	9.911495	1.49	9.850861	4.46	10.149139	39
22	762534	2.96	911405	1.49	851129	4.46	148871	38
23	762712	2.96	911315	1.50	851396	4.46	148604	37
24	762889	2.96	911226	1.50	851664	4.46	148336	36
25	763067	2.96	911136	1.50	851931	4.46	148069	35
26	763245	2.96	911046	1.50	852199	4.46	147801	34
27	763422	2.96	910956	1.50	852466	4.46	147534	33
28	763600	2.95	910866	1.50	852733	4.45	147267	32
29	763777	2.95	910776	1.50	853001	4.45	146999	31
30	763954	2.95	910686	1.50	853268	4.45	146732	30
31	9.764131	2.95	9.910596	1.50	9.853535	4.45	10.146465	29
32	764308	2.95	910506	1.50	853802	4.45	146198	28
33	764485	2.94	910415	1.50	854069	4.45	145931	27
34	764662	2.94	910325	1.51	854336	4.45	145664	26
35	764838	2.94	910235	1.51	854603	4.45	145397	25
36	765015	2.94	910144	1.51	854870	4.45	145130	24
37	765191	2.94	910054	1.51	855137	4.45	144863	23
38	765367	2.94	909963	1.51	855404	4.45	144596	22
39	765544	2.93	909873	1.51	855671	4.44	144329	21
40	765720	2.93	909782	1.51	855938	4.44	144062	20
41	9.765896	2.93	9.909691	1.51	9.856204	4.44	10.143796	19
42	766072	2.93	909601	1.51	856471	4.44	143529	18
43	766247	2.93	909510	1.51	856737	4.44	143263	17
44	766423	2.93	909419	1.51	857004	4.44	142996	16
45	766598	2.92	909328	1.52	857270	4.44	142730	15
46	766774	2.92	909237	1.52	857537	4.44	142463	14
47	766949	2.92	909146	1.52	857803	4.44	142197	13
48	767124	2.92	909055	1.52	858069	4.44	141931	12
49	767300	2.92	908964	1.52	858336	4.44	141664	11
50	767475	2.91	908873	1.52	858602	4.43	141398	10
51	9.767649	2.91	9.908781	1.52	9.858868	4.43	10.141132	9
52	767824	2.91	908690	1.52	859134	4.43	140866	8
53	767999	2.91	908599	1.52	859400	4.43	140600	7
54	768173	2.91	908507	1.52	859666	4.43	140334	6
55	768348	2.90	908416	1.53	859932	4.43	140068	5
56	768522	2.90	908324	1.53	860198	4.43	139802	4
57	768697	2.90	908233	1.53	860464	4.43	139536	3
58	768871	2.90	908141	1.53	860730	4.43	139270	2
59	769045	2.90	908049	1.53	860995	4.43	139005	1
60	769219	2.90	907958	1.53	861261	4.43	138739	0
	Cosine	D.	Sine	D.	Cotang.	D.	Tang.	M.



M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9.769219	2.90	9.907958	1.53	9.361261	4.43	138739	60
1	769393	2.89	907866	1.53	361527	4.43	138473	59
2	769566	2.89	907774	1.53	861792	4.42	138208	58
3	769740	2.89	907682	1.53	862058	4.42	137942	57
4	769913	2.89	907590	1.53	862323	4.42	137677	56
5	770087	2.89	907498	1.53	862589	4.42	137411	55
6	770260	2.88	907406	1.53	862854	4.42	137146	54
7	770433	2.88	907314	1.54	863119	4.42	136881	53
8	770606	2.88	907222	1.54	863385	4.42	136615	52
9	770779	2.88	907129	1.54	863650	4.42	136350	51
10	770952	2.88	907037	1.54	863915	4.42	136085	50
11	9.771125	2.88	9.906945	1.54	9.864180	4.42	10.135820	49
12	771298	2.87	906852	1.54	864445	4.42	135555	48
13	771470	2.87	906760	1.54	864710	4.42	135290	47
14	771643	2.87	906667	1.54	864975	4.41	135025	46
15	771815	2.87	906575	1.54	865240	4.41	134760	45
16	771987	2.87	906482	1.54	865505	4.41	134495	44
17	772159	2.87	906389	1.55	865770	4.41	134230	43
18	772331	2.86	906296	1.55	866035	4.41	133965	42
19	772503	2.86	906204	1.55	866300	4.41	133700	41
20	772675	2.86	906111	1.55	866564	4.41	133436	40
21	9.772847	2.86	9.906018	1.55	9.866829	4.41	10.133171	39
22	773018	2.86	905925	1.55	867094	4.41	132906	38
23	773190	2.86	905832	1.55	867358	4.41	132642	37
24	773361	2.85	905739	1.55	867623	4.41	132377	36
25	773533	2.85	905645	1.55	867887	4.41	132113	35
26	773704	2.85	905552	1.55	868152	4.40	131848	34
27	773875	2.85	905459	1.55	868416	4.40	131584	33
28	774046	2.85	905366	1.56	868680	4.40	131320	32
29	774217	2.85	905272	1.56	868945	4.40	131055	31
30	774388	2.84	905179	1.56	869209	4.40	130794	30
31	9.774558	2.84	9.905085	1.56	9.869473	4.40	10.130527	29
32	774729	2.84	904992	1.56	869737	4.40	130263	28
33	774899	2.84	904898	1.56	870001	4.40	129999	27
34	775070	2.84	904804	1.56	870265	4.40	129735	26
35	775240	2.84	904711	1.56	870529	4.40	129471	25
36	775410	2.83	904617	1.56	870793	4.40	129207	24
37	775580	2.83	904523	1.56	871057	4.40	128943	23
38	775750	2.83	904429	1.57	871321	4.40	128679	22
39	775920	2.83	904335	1.57	871585	4.40	128415	21
40	776090	2.83	904241	1.57	871849	4.39	128151	20
41	9.776259	2.83	9.904147	1.57	9.872112	4.39	10.127888	19
42	776429	2.82	904053	1.57	872376	4.39	127624	18
43	776598	2.82	903959	1.57	872640	4.39	127360	17
44	776768	2.82	903864	1.57	872903	4.39	127097	16
45	776937	2.82	903770	1.57	873167	4.39	126833	15
46	777106	2.82	903676	1.57	873430	4.39	126570	14
47	777275	2.81	903581	1.57	873694	4.39	126306	13
48	777444	2.81	903487	1.57	873957	4.39	126043	12
49	777613	2.81	903392	1.58	874220	4.39	125780	11
50	777781	2.81	903298	1.58	874484	4.39	125516	10
51	9.777950	2.81	9.903203	1.58	9.874747	4.39	10.125253	9
52	778119	2.81	903108	1.58	875010	4.39	124990	8
53	778287	2.80	903014	1.58	875273	4.38	124727	7
54	778455	2.80	902919	1.58	875536	4.38	124464	6
55	778624	2.80	902824	1.58	875800	4.38	124200	5
56	778792	2.80	902729	1.58	876063	4.38	123937	4
57	778960	2.80	902634	1.58	876326	4.38	123674	3
58	779128	2.80	902539	1.59	876589	4.38	123411	2
59	779295	2.79	902444	1.59	876851	4.38	123149	1
60	779463	2.79	902349	1.59	877114	4.38	122886	0
	Cosine	D.	Sine	D.	Cotang.	D.	Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9.779463	2.79	9.902349	1.59	9.877114	4.38	10.122886	60
1	779631	2.79	902253	1.59	877377	4.38	122623	59
2	779798	2.79	902158	1.59	877640	4.38	122360	58
3	779966	2.79	902063	1.59	877903	4.38	122097	57
4	780133	2.79	901967	1.59	878165	4.38	121835	56
5	780300	2.78	901872	1.59	878428	4.38	121572	55
6	780467	2.78	901776	1.59	878691	4.38	121309	54
7	780634	2.78	901681	1.59	878953	4.37	121047	53
8	780801	2.78	901585	1.59	879216	4.37	120784	52
9	780968	2.78	901490	1.59	879478	4.37	120522	51
10	781134	2.78	901394	1.60	879741	4.37	120259	50
11	9.781301	2.77	9.901298	1.60	9.880003	4.37	10.119997	49
12	781468	2.77	901202	1.60	880265	4.37	119735	48
13	781634	2.77	901106	1.60	880528	4.37	119472	47
14	781800	2.77	901010	1.60	880790	4.37	119210	46
15	781966	2.77	900914	1.60	881052	4.37	118948	45
16	782132	2.77	900818	1.60	881314	4.37	118686	44
17	782298	2.76	900722	1.60	881576	4.37	118424	43
18	782464	2.76	900626	1.60	881839	4.37	118161	42
19	782630	2.76	900529	1.60	882101	4.37	117899	41
20	782796	2.76	900433	1.61	882363	4.36	117637	40
21	9.782961	2.76	9.900337	1.61	9.882625	4.36	10.117375	39
22	783127	2.76	900240	1.61	882887	4.36	117113	38
23	783292	2.75	900144	1.61	883148	4.36	116852	37
24	783458	2.75	900047	1.61	883410	4.36	116590	36
25	783623	2.75	899951	1.61	883672	4.36	116328	35
26	783788	2.75	899854	1.61	883934	4.36	116066	34
27	783953	2.75	899757	1.61	884196	4.36	115804	33
28	784118	2.75	899660	1.61	884457	4.36	115543	32
29	784282	2.74	899564	1.61	884719	4.36	115281	31
30	784447	2.74	899467	1.62	884980	4.36	115020	30
31	9.784612	2.74	9.899370	1.62	9.885242	4.36	10.114758	29
32	784776	2.74	899273	1.62	885503	4.36	114497	28
33	784941	2.74	899176	1.62	885765	4.36	114235	27
34	785105	2.74	899078	1.62	886026	4.36	113974	26
35	785269	2.73	898981	1.62	886288	4.36	113712	25
36	785433	2.73	898884	1.62	886549	4.35	113451	24
37	785597	2.73	898787	1.62	886810	4.35	113190	23
38	785761	2.73	898689	1.62	887072	4.35	112928	22
39	785925	2.73	898592	1.62	887333	4.35	112667	21
40	786089	2.73	898494	1.63	887594	4.35	112406	20
41	9.786252	2.72	9.898397	1.63	9.887855	4.35	10.112145	19
42	786416	2.72	898299	1.63	888116	4.35	111884	18
43	786579	2.72	898202	1.63	888377	4.35	111623	17
44	786742	2.72	898104	1.63	888639	4.35	111361	16
45	786906	2.72	898006	1.63	888900	4.35	111100	15
46	787069	2.72	897908	1.63	889160	4.35	110840	14
47	787232	2.71	897810	1.63	889421	4.35	110579	13
48	787395	2.71	897712	1.63	889682	4.35	110318	12
49	787557	2.71	897614	1.63	889943	4.35	110057	11
50	787720	2.71	897516	1.63	890204	4.34	109796	10
51	9.787883	2.71	9.897418	1.64	9.890465	4.34	10.109535	9
52	788045	2.71	897320	1.64	890725	4.34	109275	8
53	788208	2.71	897222	1.64	890986	4.34	109014	7
54	788370	2.70	897123	1.64	891247	4.34	108753	6
55	788532	2.70	897025	1.64	891507	4.34	108493	5
56	788694	2.70	896926	1.64	891768	4.34	108232	4
57	788856	2.70	896828	1.64	892028	4.34	107972	3
58	789018	2.70	896729	1.64	892289	4.34	107711	2
59	789180	2.70	896631	1.64	892549	4.34	107451	1
60	789342	2.69	896532	1.64	892810	4.34	107190	0
	Cosine	D.	Sine	D.	Cotang.	D.	Tang.	M.



M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9.789342	2.69	9.896532	1.64	9.892810	4.34	10.107190	60
1	789504	2.69	896433	1.65	893070	4.34	106930	59
2	789665	2.69	896335	1.65	893331	4.34	106669	58
3	789827	2.69	896236	1.65	893591	4.34	106409	57
4	789988	2.69	896137	1.65	893851	4.34	106149	56
5	790149	2.69	896038	1.65	894111	4.34	105889	55
6	790310	2.68	895939	1.65	894371	4.34	105629	54
7	790471	2.68	895840	1.65	894632	4.33	105368	53
8	790632	2.68	895741	1.65	894892	4.33	105108	52
9	790793	2.68	895641	1.65	895152	4.33	104848	51
10	790954	2.68	895542	1.65	895412	4.33	104588	50
11	9.791115	2.68	9.895443	1.66	9.895672	4.33	10.104328	49
12	791275	2.67	895343	1.66	895932	4.33	104068	48
13	791436	2.67	895244	1.66	896192	4.33	103808	47
14	791596	2.67	895145	1.66	896452	4.33	103548	46
15	791757	2.67	895045	1.66	896712	4.33	103288	45
16	791917	2.67	894945	1.66	896971	4.33	103029	44
17	792077	2.67	894846	1.66	897231	4.33	102769	43
18	792237	2.66	894746	1.66	897491	4.33	102509	42
19	792397	2.66	894646	1.66	897751	4.33	102249	41
20	792557	2.66	894546	1.66	898010	4.33	101990	40
21	9.792716	2.66	9.894446	1.67	9.898270	4.33	10.101730	39
22	792876	2.66	894346	1.67	898530	4.33	101470	38
23	793035	2.66	894246	1.67	898789	4.33	101211	37
24	793195	2.65	894146	1.67	899049	4.32	100951	36
25	793354	2.65	894046	1.67	899308	4.32	100692	35
26	793514	2.65	893946	1.67	899568	4.32	100432	34
27	793673	2.65	893846	1.67	899827	4.32	100172	33
28	793832	2.65	893745	1.67	900086	4.32	999914	32
29	793991	2.65	893645	1.67	900346	4.32	999654	31
30	794150	2.64	893544	1.67	900605	4.32	999395	30
31	9.794308	2.64	9.893444	1.68	9.900864	4.32	10.099136	29
32	794467	2.64	893343	1.68	901124	4.32	998876	28
33	794626	2.64	893243	1.68	901383	4.32	998617	27
34	794784	2.64	893142	1.68	901642	4.32	998358	26
35	794942	2.64	893041	1.68	901901	4.32	998099	25
36	795101	2.64	892940	1.68	902160	4.32	997840	24
37	795259	2.63	892839	1.68	902419	4.32	997581	23
38	795417	2.63	892739	1.68	902679	4.32	997321	22
39	795575	2.63	892638	1.68	902938	4.32	997062	21
40	795733	2.63	892536	1.68	903197	4.31	996803	20
41	9.795891	2.63	9.892435	1.69	9.903455	4.31	10.096545	19
42	796049	2.63	892334	1.69	903714	4.31	996286	18
43	796206	2.63	892233	1.69	903973	4.31	996027	17
44	796364	2.62	892132	1.69	904232	4.31	995768	16
45	796521	2.62	892030	1.69	904491	4.31	995509	15
46	796679	2.62	891929	1.69	904750	4.31	995250	14
47	796836	2.62	891827	1.69	905008	4.31	994992	13
48	796993	2.62	891726	1.69	905267	4.31	994733	12
49	797150	2.61	891624	1.69	905526	4.31	994474	11
50	797307	2.61	891523	1.70	905784	4.31	994216	10
51	9.797464	2.61	9.891421	1.70	9.906043	4.31	10.093957	9
52	797621	2.61	891319	1.70	906302	4.31	993698	8
53	797777	2.61	891217	1.70	906560	4.31	993440	7
54	797934	2.61	891115	1.70	906819	4.31	993181	6
55	798091	2.61	891013	1.70	907077	4.31	992923	5
56	798247	2.61	890911	1.70	907336	4.31	992664	4
57	798403	2.60	890809	1.70	907594	4.31	992406	3
58	798560	2.60	890707	1.70	907852	4.31	992148	2
59	798716	2.60	890605	1.70	908111	4.30	991889	1
60	798872	2.60	890503	1.70	908369	4.30	991631	0
	Cosine	D.	Sine	D.	Cotang.	D.	Tang.	M.



M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9.798872	2.60	9.890503	1.70	9.908369	4.30	10.091631	60
1	799c28	2.60	890400	1.71	908628	4.30	091372	59
2	799184	2.60	890298	1.71	908886	4.30	091114	58
3	799339	2.59	890195	1.71	909144	4.30	090856	57
4	799495	2.59	890093	1.71	909402	4.30	090598	56
5	799651	2.59	889990	1.71	909660	4.30	090340	55
6	799806	2.59	889888	1.71	909918	4.30	090082	54
7	799962	2.59	889785	1.71	910177	4.30	089823	53
8	800117	2.59	889682	1.71	910435	4.30	089565	52
9	800272	2.58	889579	1.71	910693	4.30	089307	51
10	800427	2.58	889477	1.71	910951	4.30	089049	50
11	9.800582	2.58	9.889374	1.72	9.911209	4.30	10.088791	49
12	800737	2.58	889271	1.72	911467	4.30	088533	48
13	800892	2.58	889168	1.72	911724	4.30	088276	47
14	801047	2.58	889064	1.72	911982	4.30	088018	46
15	801201	2.58	888961	1.72	912240	4.30	087760	45
16	801356	2.57	888858	1.72	912498	4.30	087502	44
17	801511	2.57	888755	1.72	912756	4.30	087244	43
18	801665	2.57	888651	1.72	913014	4.29	086986	42
19	801819	2.57	888548	1.72	913271	4.29	086729	41
20	801973	2.57	888444	1.73	913529	4.29	086471	40
21	9.802128	2.57	9.888341	1.73	9.913787	4.29	10.086213	39
22	802282	2.56	888237	1.73	914044	4.29	085956	38
23	802436	2.56	888134	1.73	914302	4.29	085698	37
24	802589	2.56	888030	1.73	914560	4.29	085440	36
25	802743	2.56	887926	1.73	914817	4.29	085183	35
26	802897	2.56	887822	1.73	915075	4.29	084925	34
27	803050	2.56	887718	1.73	915332	4.29	084668	33
28	803204	2.56	887614	1.73	915590	4.29	084410	32
29	803357	2.55	887510	1.73	915847	4.29	084153	31
30	803511	2.55	887406	1.74	916104	4.29	083896	30
31	9.803664	2.55	9.887302	1.74	9.916362	4.29	10.083638	29
32	803817	2.55	887198	1.74	916619	4.29	083381	28
33	803970	2.55	887093	1.74	916877	4.29	083123	27
34	804123	2.55	886989	1.74	917134	4.29	082866	26
35	804276	2.54	886885	1.74	917391	4.29	082609	25
36	804428	2.54	886780	1.74	917648	4.29	082352	24
37	804581	2.54	886676	1.74	917905	4.29	082095	23
38	804734	2.54	886571	1.74	918163	4.28	081837	22
39	804886	2.54	886466	1.74	918420	4.28	081580	21
40	805039	2.54	886362	1.75	918677	4.28	081323	20
41	9.805191	2.54	9.886257	1.75	9.918934	4.28	10.081066	19
42	805343	2.53	886152	1.75	919191	4.28	080809	18
43	805495	2.53	886047	1.75	919448	4.28	080552	17
44	805647	2.53	885942	1.75	919705	4.28	080295	16
45	805799	2.53	885837	1.75	919962	4.28	080038	15
46	805951	2.53	885732	1.75	920219	4.28	079781	14
47	806103	2.53	885627	1.75	920476	4.28	079524	13
48	806254	2.53	885522	1.75	920733	4.28	079267	12
49	806406	2.52	885416	1.75	920990	4.28	079010	11
50	806557	2.52	885311	1.76	921247	4.28	078753	10
51	9.806709	2.52	9.885205	1.76	9.921503	4.28	10.078497	9
52	806860	2.52	885100	1.76	921760	4.28	078240	8
53	807011	2.52	884994	1.76	922017	4.28	077983	7
54	807163	2.52	884889	1.76	922274	4.28	077726	6
55	807314	2.52	884783	1.76	922530	4.28	077470	5
56	807465	2.51	884677	1.76	922787	4.28	077213	4
57	807615	2.51	884572	1.76	923044	4.28	076956	3
58	807766	2.51	884466	1.76	923300	4.28	076700	2
59	807917	2.51	884360	1.76	923557	4.27	076443	1
60	808067	2.51	884254	1.77	923813	4.27	076187	0
	Cosine	D.	Sine	D.	Cotang.	D.	Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9.808067	2.51	9.884254	1.77	9.923813	4.27	10.076187	60
1	808218	2.51	884148	1.77	924070	4.27	075930	59
2	808368	2.51	884042	1.77	924327	4.27	075673	58
3	808519	2.50	883936	1.77	924583	4.27	075417	57
4	808669	2.50	883829	1.77	924840	4.27	075160	56
5	808817	2.50	883723	1.77	925096	4.27	074904	55
6	808969	2.50	883617	1.77	925352	4.27	074648	54
7	809119	2.50	883510	1.77	925609	4.27	074391	53
8	809269	2.50	883404	1.77	925865	4.27	074135	52
9	809419	2.49	883297	1.78	926122	4.27	073878	51
10	809569	2.49	883191	1.78	926378	4.27	073622	50
11	9.809718	2.49	9.883084	1.78	9.926634	4.27	10.073366	49
12	809868	2.49	882977	1.78	926890	4.27	073110	48
13	810017	2.49	882871	1.78	927147	4.27	072853	47
14	810167	2.49	882764	1.78	927403	4.27	072597	46
15	810316	2.48	882657	1.78	927659	4.27	072341	45
16	810465	2.48	882550	1.78	927915	4.27	072085	44
17	810614	2.48	882443	1.78	928171	4.27	071829	43
18	810763	2.48	882336	1.79	928427	4.27	071573	42
19	810912	2.48	882229	1.79	928683	4.27	071317	41
20	811061	2.48	882121	1.79	928940	4.27	071060	40
21	9.811210	2.48	9.882014	1.79	9.929196	4.27	10.070804	39
22	811358	2.47	881907	1.79	929452	4.27	070548	38
23	811507	2.47	881799	1.79	929708	4.27	070292	37
24	811655	2.47	881692	1.79	929964	4.26	070036	36
25	811804	2.47	881584	1.79	930220	4.26	069780	35
26	811952	2.47	881477	1.79	930475	4.26	069525	34
27	812100	2.47	881369	1.79	930731	4.26	069269	33
28	812248	2.47	881261	1.80	930987	4.26	069013	32
29	812396	2.46	881153	1.80	931243	4.26	068757	31
30	812544	2.46	881046	1.80	931499	4.26	068501	30
31	9.812692	2.46	9.880938	1.80	9.931755	4.26	10.068245	29
32	812840	2.46	880830	1.80	932010	4.26	067990	28
33	812988	2.46	880722	1.80	932266	4.26	067734	27
34	813135	2.46	880613	1.80	932522	4.26	067478	26
35	813283	2.46	880505	1.80	932778	4.26	067222	25
36	813430	2.45	880397	1.80	933033	4.26	066967	24
37	813578	2.45	880289	1.81	933289	4.26	066711	23
38	813725	2.45	880180	1.81	933545	4.26	066455	22
39	813872	2.45	880072	1.81	933800	4.26	066200	21
40	814019	2.45	879963	1.81	934056	4.26	065944	20
41	9.814166	2.45	9.879855	1.81	9.934311	4.26	10.065689	19
42	814313	2.45	879746	1.81	934567	4.26	065433	18
43	814460	2.44	879637	1.81	934823	4.26	065177	17
44	814607	2.44	879529	1.81	935078	4.26	064922	16
45	814753	2.44	879420	1.81	935333	4.26	064667	15
46	814900	2.44	879311	1.81	935589	4.26	064411	14
47	815046	2.44	879202	1.82	935844	4.26	064156	13
48	815193	2.44	879093	1.82	936100	4.26	063900	12
49	815339	2.44	878984	1.82	936355	4.26	063645	11
50	815485	2.43	878875	1.82	936610	4.26	063390	10
51	9.815631	2.43	9.878766	1.82	9.936866	4.25	10.063134	9
52	815778	2.43	878656	1.82	937121	4.25	062879	8
53	815924	2.43	878547	1.82	937376	4.25	062624	7
54	816069	2.43	878438	1.82	937631	4.25	062368	6
55	816215	2.43	878328	1.82	937887	4.25	062113	5
56	816361	2.43	878219	1.83	938143	4.25	061858	4
57	816507	2.42	878109	1.83	938398	4.25	061602	3
58	816652	2.42	877999	1.83	938653	4.25	061347	2
59	816798	2.42	877890	1.83	938908	4.25	061092	1
60	816943	2.42	877780	1.83	939163	4.25	060837	0
	Cosine	D.	Sine	D.	Cotang.	D.	Tang.	



M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9.816943	2.42	9.877780	.83	9.939163	4.25	10.060837	60
1	817088	2.42	877670	1.83	939418	4.25	060582	59
2	817233	2.42	877560	1.83	939673	4.25	060327	58
3	817379	2.42	877450	1.83	939928	4.25	060072	57
4	817524	2.41	877340	1.83	940183	4.25	059817	56
5	817668	2.41	877230	1.84	940438	4.25	059562	55
6	817813	2.41	877120	1.84	940694	4.25	059306	54
7	817958	2.41	877010	1.84	940949	4.25	059051	53
8	818103	2.41	876899	1.84	941204	4.25	058796	52
9	818247	2.41	876789	1.84	941458	4.25	058542	51
10	818392	2.41	876678	1.84	941714	4.25	058286	50
11	9.818536	2.40	9.876568	1.84	9.941968	4.25	10.058032	49
12	818681	2.40	876457	1.84	942223	4.25	057777	48
13	818825	2.40	876347	1.84	942478	4.25	057522	47
14	818969	2.40	876236	1.85	942733	4.25	057267	46
15	819113	2.40	876125	1.85	942988	4.25	057012	45
16	819257	2.40	876014	1.85	943243	4.25	056757	44
17	819401	2.40	875904	1.85	943498	4.25	056502	43
18	819545	2.39	875793	1.85	943752	4.25	056248	42
19	819689	2.39	875682	1.85	944007	4.25	055993	41
20	819832	2.39	875571	1.85	944262	4.25	055738	40
21	9.819976	2.39	9.875459	1.85	9.944517	4.25	10.055483	39
22	820120	2.39	875348	1.85	944771	4.24	055229	38
23	820263	2.39	875237	1.85	945026	4.24	054974	37
24	820406	2.39	875126	1.86	945281	4.24	054719	36
25	820550	2.38	875014	1.86	945535	4.24	054465	35
26	820693	2.38	874903	1.86	945790	4.24	054210	34
27	820836	2.38	874791	1.86	946045	4.24	053955	33
28	820979	2.38	874680	1.86	946299	4.24	053701	32
29	821122	2.38	874568	1.86	946554	4.24	053446	31
30	821265	2.38	874456	1.86	946808	4.24	053192	30
31	9.821407	2.38	9.874344	1.86	9.947063	4.24	10.052937	29
32	821550	2.38	874232	1.87	947318	4.24	052682	28
33	821693	2.37	874121	1.87	947572	4.24	052428	27
34	821835	2.37	874009	1.87	947826	4.24	052174	26
35	821977	2.37	873896	1.87	948081	4.24	051919	25
36	822120	2.37	873784	1.87	948336	4.24	051664	24
37	822262	2.37	873672	1.87	948590	4.24	051410	23
38	822404	2.37	873560	1.87	948844	4.24	051156	22
39	822546	2.37	873448	1.87	949099	4.24	050901	21
40	822688	2.36	873335	1.87	949353	4.24	050647	20
41	9.822830	2.36	9.873223	1.87	9.949607	4.24	10.050393	19
42	822972	2.36	873110	1.88	949862	4.24	050138	18
43	823114	2.36	872998	1.88	950116	4.24	049884	17
44	823255	2.36	872885	1.88	950370	4.24	049630	16
45	823397	2.36	872772	1.88	950625	4.24	049375	15
46	823539	2.36	872659	1.88	950879	4.24	049121	14
47	823680	2.35	872547	1.88	951133	4.24	048867	13
48	823821	2.35	872434	1.88	951388	4.24	048612	12
49	823963	2.35	872321	1.88	951642	4.24	048358	11
50	824104	2.35	872208	1.88	951896	4.24	048104	10
51	9.824245	2.35	9.872095	1.89	9.952150	4.24	10.047850	9
52	824386	2.35	871981	1.89	952405	4.24	047595	8
53	824527	2.35	871868	1.89	952659	4.24	047341	7
54	824668	2.34	871755	1.89	952913	4.24	047087	6
55	824808	2.34	871641	1.89	953167	4.23	046833	5
56	824949	2.34	871528	1.89	953421	4.23	046579	4
57	825090	2.34	871414	1.89	953675	4.23	046325	3
58	825230	2.34	871301	1.89	953929	4.23	046071	2
59	825371	2.34	871187	1.89	954183	4.23	045817	1
60	825511	2.34	871073	1.90	954437	4.23	045563	0
	Cosine	D.	Sine	D.	Cotang.	D.	Tang.	M.



M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9.825511	2.34	9.871073	1.90	9.954437	4.23	10.045563	60
1	825651	2.33	870960	1.90	954691	4.23	045309	59
2	825791	2.33	870846	1.90	954945	4.23	045055	58
3	825931	2.33	870732	1.90	955200	4.23	044800	57
4	826071	2.33	870618	1.90	955454	4.23	044546	56
5	826211	2.33	870504	1.90	955707	4.23	044293	55
6	826351	2.33	870390	1.90	955961	4.23	044039	54
7	826491	2.33	870276	1.90	956215	4.23	043785	53
8	826631	2.33	870161	1.90	956469	4.23	043531	52
9	826770	2.32	870047	1.91	956723	4.23	043277	51
10	826910	2.32	869933	1.91	956977	4.23	043023	50
11	9.827049	2.32	9.869818	1.91	9.957231	4.23	10.042769	49
12	827189	2.32	869704	1.91	957485	4.23	042515	48
13	827328	2.32	869589	1.91	957739	4.23	042261	47
14	827467	2.32	869474	1.91	957993	4.23	042007	46
15	827606	2.32	869360	1.91	958246	4.23	041754	45
16	827745	2.32	869245	1.91	958500	4.23	041500	44
17	827884	2.31	869130	1.91	958754	4.23	041246	43
18	828023	2.31	869015	1.92	959008	4.23	040992	42
19	828162	2.31	868900	1.92	959262	4.23	040738	41
20	828301	2.31	868785	1.92	959516	4.23	040484	40
21	9.828439	2.31	9.868670	1.92	9.959769	4.23	10.040231	39
22	828578	2.31	868555	1.92	960023	4.23	039977	38
23	828716	2.31	868440	1.92	960277	4.23	039723	37
24	828855	2.30	868324	1.92	960531	4.23	039469	36
25	828993	2.30	868209	1.92	960784	4.23	039216	35
26	829131	2.30	868093	1.92	961038	4.23	038962	34
27	829269	2.30	867978	1.93	961291	4.23	038709	33
28	829407	2.30	867862	1.93	961545	4.23	038455	32
29	829545	2.30	867747	1.93	961799	4.23	038201	31
30	829683	2.30	867631	1.93	962052	4.23	037948	30
31	9.829821	2.29	9.867515	1.93	9.962306	4.23	10.037694	29
32	829959	2.29	867399	1.93	962560	4.23	037440	28
33	830097	2.29	867283	1.93	962813	4.23	037187	27
34	830234	2.29	867167	1.93	963067	4.23	036933	26
35	830372	2.29	867051	1.93	963320	4.23	036680	25
36	830509	2.29	866935	1.94	963574	4.23	036426	24
37	830646	2.29	866819	1.94	963827	4.23	036173	23
38	830784	2.29	866703	1.94	964081	4.23	035919	22
39	830921	2.28	866586	1.94	964335	4.23	035665	21
40	831058	2.28	866470	1.94	964588	4.22	035412	20
41	9.831195	2.28	9.866353	1.94	9.964842	4.22	10.035158	19
42	831332	2.28	866237	1.94	965095	4.22	034905	18
43	831469	2.28	866120	1.94	965349	4.22	034651	17
44	831606	2.28	866004	1.95	965602	4.22	034398	16
45	831742	2.28	865887	1.95	965855	4.22	034145	15
46	831879	2.28	865770	1.95	966105	4.22	033891	14
47	832015	2.27	865653	1.95	966362	4.22	033638	13
48	832152	2.27	865536	1.95	966616	4.22	033384	12
49	832288	2.27	865419	1.95	966869	4.22	033131	11
50	832425	2.27	865302	1.95	967123	4.22	032877	10
51	9.832561	2.27	9.865185	1.95	9.967376	4.22	10.032624	9
52	832697	2.27	865068	1.95	967629	4.22	032371	8
53	832833	2.27	864950	1.95	967883	4.22	032117	7
54	832969	2.26	864833	1.96	968136	4.22	031864	6
55	833105	2.26	864716	1.96	968389	4.22	031611	5
56	833241	2.26	864598	1.96	968643	4.22	031357	4
57	833377	2.26	864481	1.96	968896	4.22	031104	3
58	833512	2.26	864363	1.96	969149	4.22	030851	2
59	833648	2.26	864245	1.96	969403	4.22	030597	1
60	833783	2.26	864127	1.96	969656	4.22	030344	0
	Cosine	D.	Sine	D.	Cotang.	D.	Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9.833783	2.26	9.864127	1.96	9.969656	4.22	10.030344	50
1	833919	2.25	864010	1.96	969909	4.22	030091	59
2	834054	2.25	863892	1.97	970162	4.22	029838	58
3	834189	2.25	863774	1.97	970416	4.22	029584	57
4	834325	2.25	863656	1.97	970669	4.22	029331	56
5	834460	2.25	863538	1.97	970922	4.22	029078	55
6	834595	2.25	863419	1.97	971175	4.22	028825	54
7	834730	2.25	863301	1.97	971429	4.22	028571	53
8	834865	2.25	863183	1.97	971682	4.22	028318	52
9	834999	2.24	863064	1.97	971935	4.22	028065	51
10	835134	2.24	862946	1.98	972188	4.22	027812	50
11	9.835269	2.24	9.862827	1.98	9.972441	4.22	10.027559	49
12	835403	2.24	862709	1.98	972694	4.22	027306	48
13	835538	2.24	862590	1.98	972948	4.22	027052	47
14	835672	2.24	862471	1.98	973201	4.22	026799	46
15	835807	2.24	862353	1.98	973454	4.22	026546	45
16	835941	2.24	862234	1.98	973707	4.22	026293	44
17	836075	2.23	862115	1.98	973960	4.22	026040	43
18	836209	2.23	861996	1.98	974213	4.22	025787	42
19	836343	2.23	861877	1.98	974466	4.22	025534	41
20	836477	2.23	861758	1.99	974719	4.22	025281	40
21	9.836611	2.23	9.861638	1.99	9.974973	4.22	10.025027	39
22	836745	2.23	861519	1.99	975226	4.22	024774	38
23	836878	2.23	861400	1.99	975479	4.22	024521	37
24	837012	2.22	861280	1.99	975732	4.22	024268	36
25	837146	2.22	861161	1.99	975985	4.22	024015	35
26	837279	2.22	861041	1.99	976238	4.22	023762	34
27	837412	2.22	860922	1.99	976491	4.22	023509	33
28	837546	2.22	860802	1.99	976744	4.22	023256	32
29	837679	2.22	860682	2.00	976997	4.22	023003	31
30	837812	2.22	860562	2.00	977250	4.22	022750	30
31	9.837945	2.22	9.860442	2.00	9.977503	4.22	10.022497	29
32	838078	2.21	860322	2.00	977756	4.22	022244	28
33	838211	2.21	860202	2.00	978009	4.22	021991	27
34	838344	2.21	860082	2.00	978262	4.22	021738	26
35	838477	2.21	859962	2.00	978515	4.22	021485	25
36	838610	2.21	859842	2.00	978768	4.22	021232	24
37	838742	2.21	859721	2.01	979021	4.22	020979	23
38	838875	2.21	859601	2.01	979274	4.22	020726	22
39	839007	2.21	859480	2.01	979527	4.22	020473	21
40	839140	2.20	859360	2.01	979780	4.22	020220	20
41	9.839272	2.20	9.859239	2.01	9.980033	4.22	10.019967	19
42	839404	2.20	859119	2.01	980286	4.22	019714	18
43	839536	2.20	858998	2.01	980538	4.22	019462	17
44	839668	2.20	858877	2.01	980791	4.21	019209	16
45	839800	2.20	858756	2.02	981044	4.21	018956	15
46	839932	2.20	858635	2.02	981297	4.21	018703	14
47	840064	2.19	858514	2.02	981550	4.21	018450	13
48	840196	2.19	858393	2.02	981803	4.21	018197	12
49	840328	2.19	858272	2.02	982056	4.21	017944	11
50	840459	2.19	858151	2.02	982309	4.21	017691	10
51	9.840591	2.19	9.858029	2.02	9.982562	4.21	10.017438	9
52	840722	2.19	857908	2.02	982814	4.21	017186	8
53	840854	2.19	857786	2.02	983067	4.21	016933	7
54	840985	2.19	857665	2.03	983320	4.21	016680	6
55	841116	2.18	857543	2.03	983573	4.21	016427	5
56	841247	2.18	857422	2.03	983826	4.21	016174	4
57	841378	2.18	857300	2.03	984079	4.21	015921	3
58	841509	2.18	857178	2.03	984331	4.21	015669	2
59	841640	2.18	857056	2.03	984584	4.21	015416	1
60	841771	2.18	856934	2.03	984837	4.21	015163	0
	Cosine	D.	Sine	D.	Cotang	D.	Tang.	M.



M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9.841771	2.18	9.856934	2.03	9.984837	4.21	10.015163	60
1	841902	2.18	856812	2.03	985090	4.21	014910	59
2	842033	2.18	856690	2.04	985343	4.21	014657	58
3	842163	2.17	856568	2.04	985596	4.21	014404	57
4	842294	2.17	856446	2.04	985848	4.21	014152	56
5	842424	2.17	856323	2.04	986101	4.21	013899	55
6	842555	2.17	856201	2.04	986354	4.21	013646	54
7	842685	2.17	856078	2.04	986607	4.21	013393	53
8	842815	2.17	855956	2.04	986860	4.21	013140	52
9	842946	2.17	855833	2.04	987112	4.21	012888	51
10	843076	2.17	855711	2.05	987365	4.21	012635	50
11	9.843206	2.16	9.855588	2.05	9.987618	4.21	10.012382	49
12	843336	2.16	855465	2.05	987871	4.21	012129	48
13	843466	2.16	855342	2.05	988123	4.21	011877	47
14	843595	2.16	855219	2.05	988376	4.21	011624	46
15	843725	2.16	855096	2.05	988629	4.21	011371	45
16	843855	2.16	854973	2.05	988882	4.21	011118	44
17	843984	2.16	854850	2.05	989134	4.21	010866	43
18	844114	2.15	854727	2.06	989387	4.21	010613	42
19	844243	2.15	854603	2.06	989640	4.21	010360	41
20	844372	2.15	854480	2.06	989893	4.21	010107	40
21	9.844502	2.15	9.854356	2.06	9.990145	4.21	10.009855	39
22	844631	2.15	854233	2.06	990398	4.21	009602	38
23	844760	2.15	854109	2.06	990651	4.21	009349	37
24	844889	2.15	853986	2.06	990903	4.21	009097	36
25	845018	2.15	853862	2.06	991156	4.21	008844	35
26	845147	2.15	853738	2.06	991409	4.21	008591	34
27	845276	2.14	853614	2.07	991662	4.21	008338	33
28	845405	2.14	853490	2.07	991914	4.21	008086	32
29	845533	2.14	853366	2.07	992167	4.21	007833	31
30	845662	2.14	853242	2.07	992420	4.21	007580	30
31	9.845790	2.14	9.853118	2.07	9.992672	4.21	10.007328	29
32	845919	2.14	852994	2.07	992925	4.21	007075	28
33	846047	2.14	852869	2.07	993178	4.21	006822	27
34	846175	2.14	852745	2.07	993430	4.21	006570	26
35	846304	2.14	852620	2.07	993683	4.21	006317	25
36	846432	2.13	852496	2.08	993936	4.21	006064	24
37	846560	2.13	852371	2.08	994189	4.21	005811	23
38	846688	2.13	852247	2.08	994441	4.21	005559	22
39	846816	2.13	852122	2.08	994694	4.21	005306	21
40	846944	2.13	851997	2.08	994947	4.21	005053	20
41	9.847071	2.13	9.851872	2.08	9.995199	4.21	10.004801	19
42	847199	2.13	851747	2.08	995452	4.21	004548	18
43	847327	2.13	851622	2.08	995705	4.21	004295	17
44	847454	2.12	851497	2.09	995957	4.21	004043	16
45	847582	2.12	851372	2.09	996210	4.21	003790	15
46	847709	2.12	851246	2.09	996463	4.21	003537	14
47	847836	2.12	851121	2.09	996715	4.21	003285	13
48	847964	2.12	850996	2.09	996968	4.21	003032	12
49	848091	2.12	850870	2.09	997221	4.21	002779	11
50	848218	2.12	850745	2.09	997473	4.21	002527	10
51	9.848345	2.12	9.850619	2.09	9.997726	4.21	10.002274	9
52	848472	2.11	850493	2.10	997979	4.21	002021	8
53	848599	2.11	850368	2.10	998231	4.21	001769	7
54	848726	2.11	850242	2.10	998484	4.21	001516	6
55	848852	2.11	850116	2.10	998737	4.21	001263	5
56	848979	2.11	849990	2.10	998989	4.21	001011	4
57	849106	2.11	849864	2.10	999242	4.21	000758	3
58	849232	2.11	849738	2.10	999495	4.21	000505	2
59	849359	2.11	849611	2.10	999748	4.21	000253	1
60	849485	2.11	849485	2.10	10.000000	4.21	10.000000	0
	Cosine	D.	Sine	D.	Cotang.	D.	Tang.	M.







JRSI  
1985



